Applications of Exponentials and Logarithms

Exponential Functions and Interest

In this section we derive some formulas which are used to calculate how much interest an investment earns.

Some terminology

1. Interest refers to the amount of money paid for the use of money. It may also refer to the rate of such payment. For example, if a bank lends you $100 dollars, and at the end of one week you have to pay it back plus an additional $10 dollars, then the bank has earned $10 dollars in interest. We could also say that the bank’s interest rate on this loan is 10% per week. $\frac{10}{100} = \frac{1}{10}$ is the ratio of interest to amount borrowed, and in terms of percentages equals $\frac{10}{100} \times 100 = 10\%$ per week.

2. Simple interest means that interest on an investment is paid at the end of the interest period.

3. Compound interest means that interest is paid in installments during the interest period, and then these interest payments also earn interest. Most of the examples we study will involve compound interest.

An easily remembered formula for computing the interest rate in percentages is the following

\[
\text{interest rate (percentage)} = \frac{\text{interest earned}}{\text{amount borrowed}} \times 100
\]

Example 1: If a bank charges you 6.75% interest for 1 year on a loan of $250 dollars, how much money will you have to pay the bank at the end of the year?

Solution: You will have to pay back the original $250 plus the interest. The interest equals 6.75% of $250.

\[
\text{Pay to bank} = 250 + 0.0675(250) \quad \text{note the conversion of}
\]

\[
= 250 + 16.875 \quad 6.75\% \to 0.0675
\]

\[
\approx 266.88 \text{ dollars.}
\]

Example 2: If you borrow $50 from your brother, and at the end of 1 year pay back $56, what is the interest your brother has earned? What is the interest rate as a percentage?

Solution: The interest earned is the difference between what was borrowed and what was paid back. Thus, the interest equals $56 - 50 = 6$ dollars. The interest rate equals

\[
\text{interest rate} = \frac{\text{interest}}{\text{amount borrowed}} = \frac{6}{50} = 0.12.
\]

As a percentage the interest rate equals $(0.12)100 = 12\%$ per year.

Example 3: If your mother borrows $1000 from a bank, and pays it back after one year plus an interest of $126, what is the interest rate as a percentage?

Answer: The interest rate is $\frac{126}{1000} = 0.126$. Converting to a percentage, this is equivalent to $12.6\%$.

Example 4: Suppose we invest $100 at 5\%$ per year for one year. How much money do we have at the end of the year?

Solution: The answer is arrived at as follows. After 1 year the amount of interest earned is 5\% of 100 or $(0.05)100 = 5.0$. Thus, the total amount of money (original investment plus interest earned) equals $105.00$.
Example 5: You have a savings account with 100 dollars and the bank pays you 5% simple interest per year. How much money will you have in the savings account after 2 years?

Solution: You will have a grand total of $110.25. Since the bank is paying interest at a rate of 5% per year, after 1 year you will have $100 + 0.05(100) = 105. After the second year you will have the $105 plus the interest it earns, (0.05)105. Thus, you will have $105 + 0.05(105) = 110.25. The extra 25¢ comes from the interest on the $5 interest earned the first year.

Example 6: You have a savings account with $100, and the bank pays you 5% interest compounded twice per year. How much money will you have in the savings account after 2 years?

Solution: Compounding twice per year is a complicated way to say that the bank’s rate of interest is \( \frac{5}{2} \) % per half year. So over a 2 year time frame there are 4 interest periods, with each interest period of length 1/2 year. This means that there are 4 interest periods in the 2 years. The table below shows how the money grows.

<table>
<thead>
<tr>
<th>Interest Period</th>
<th>$ at start</th>
<th>$ at end</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>100</td>
<td>105</td>
</tr>
<tr>
<td>2</td>
<td>105.025</td>
<td>107.68906</td>
</tr>
<tr>
<td>3</td>
<td>107.68906</td>
<td>110.38129</td>
</tr>
</tbody>
</table>

So after 2 years the savings account has earned $10.38. This is 13¢ more than the same account where there was no compounding.

We now discuss some examples involving compound interest. The problem we will look at is the following: Suppose you deposit money in an account which pays \( r \) percent interest per year with interest compounded \( k \) times per year. How much money will you have in the account after \( t \) years. The answer in terms of a formula is given below:

Let \( P \) denote the amount of money originally invested. Then, at an interest rate of \( r \) percent compounded \( k \) times per year, the amount of money available after \( t \) years equals

\[
A(t) = P \left(1 + \frac{r}{100 \cdot k}\right)^{kt} ,
\]

where \( r \) in the formula is a percent. That is, if the interest rate is 5%, then \( r = 5 \).

Proof of Formula for Compound Interest: Since the interest is compounded \( k \) times per year this means in a single interest period that the interest rate is \( \frac{r}{k} \). The table below shows the amount of money after \( i \) interest periods.

<table>
<thead>
<tr>
<th>Interest Period</th>
<th>Amount at Start</th>
<th>Amount at End</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( P )</td>
<td>( P \left(1 + \frac{r}{k}\right) )</td>
</tr>
<tr>
<td>2</td>
<td>( P \left(1 + \frac{r}{k}\right) )</td>
<td>( P \left(1 + \frac{r}{k}\right) \left(1 + \frac{r}{k}\right) )</td>
</tr>
<tr>
<td>3</td>
<td>( P \left(1 + \frac{r}{k}\right)^2 )</td>
<td>( P \left(1 + \frac{r}{k}\right)^2 \left(1 + \frac{r}{k}\right) )</td>
</tr>
<tr>
<td>4</td>
<td>( P \left(1 + \frac{r}{k}\right)^3 )</td>
<td>( P \left(1 + \frac{r}{k}\right)^3 \left(1 + \frac{r}{k}\right) )</td>
</tr>
<tr>
<td>( \vdots )</td>
<td>( \vdots )</td>
<td>( \vdots )</td>
</tr>
<tr>
<td>1 year or ( k ) periods</td>
<td>( P \left(1 + \frac{r}{k}\right)^{k-1} )</td>
<td>( P \left(1 + \frac{r}{k}\right)^{k-1} \left(1 + \frac{r}{k}\right) )</td>
</tr>
<tr>
<td>( t^{th} ) year or ( tk ) periods</td>
<td>( P \left(1 + \frac{r}{k}\right)^{dk-1} )</td>
<td>( P \left(1 + \frac{r}{k}\right)^{dk-1} \left(1 + \frac{r}{k}\right) )</td>
</tr>
</tbody>
</table>
An explanation for the entries is as follows. Let’s take line 2. This represents the second interest period. At the start of the second interest period we have $A \left( \frac{1}{k} \right) = P \left( 1 + \frac{r}{k} \right)$. So at the end of the second interest period we have the amount at the start of the second interest period plus the interest it earns. That is,

$$A \left( \frac{2}{k} \right) = P \left( 1 + \frac{r}{k} \right) + \frac{r}{k} P \left( 1 + \frac{r}{k} \right) = P \left( 1 + \frac{r}{k} \right) \left( 1 + r \frac{k}{k} \right) = P \left( 1 + \frac{r}{k} \right)^2.$$

Example 7: Suppose you invest $100, and it earns 6% compounded quarterly. How much money will you have after 5 years?

Solution: $P = 100, k = 4, t = 5, and r = 6$. Thus the amount of money after 5 years equals

$$A(5) = 100 \left( 1 + \frac{6}{100 \times 4} \right)^{4 \times 5} = 100 \left( 1.015 \right)^{20} = 100 \left( 1.015 \right)^{20} \approx 100 \left( 1.346855 \right) \approx 134.69.$$

Question: Suppose you had two investment possibilities, both with the same interest rate $r$, but one of them is compounded more often than the other. That is, suppose $k_1$ and $k_2$ denote the number of times the investments are compounded per year, and $k_1 < k_2$, which investment earns more money?

Answer: Given that both investments have the same interest rate, then the investment that is compounded more often grows faster. Not by much it is true, but the more interest is compounded the more an investment appreciates, i.e., grows.

Example 8: You invest $500 at 6% compounded monthly. How much money will you have after 10 years?

Solution: To solve this problem we use the formula

$$A(t) = P \left( 1 + \frac{r}{100 \times k} \right)^{kt},$$

where $P = 500, r = 6, k = 12, and t = 10.$

Amount after 10 years

$$= 500 \left( 1 + \frac{6}{1200} \right)^{12 \times 10}$$

$$= 500 \left( 1.005 \right)^{120}$$

$$\approx 909.69.$$

After 10 years the original $500 will be worth approximately $910.

Example 9: Suppose you can invest money at 8% compounded monthly. How much money would you have to invest in order to have $20,000 after 15 years?

Solution: In this example we are given $A(15)$ and asked to solve for the original investment $P$. The formula relating these quantities is the same one we have been using. That is,

$$20,000 = A(15) = P \left( 1 + \frac{8}{1200} \right)^{12 \times 15}.$$

We merely need to solve this equation for $P$.

$$20,000 = P \left( 1.0066666667 \right)^{180}$$

$$\approx P \left( 3.3069215 \right).$$

Thus, we have

$$P \approx \frac{20,000}{3.3069215}$$

$$\approx 6048.$$

Thus, an original investment of about $6048 at 8% compounded monthly will be worth $20,000 after 10 years.
In this page we examine what happens when the number of interest periods per year starts to increase. The table below shows how much money we’ll have if $100 is invested at 5% compounded 10, 20, 50, and 100 times per year for 25 years and 50 years.

<table>
<thead>
<tr>
<th></th>
<th>$k = 10$</th>
<th>$k = 20$</th>
<th>$k = 50$</th>
<th>$k = 100$</th>
</tr>
</thead>
<tbody>
<tr>
<td>25 years at 5%</td>
<td>347.95</td>
<td>348.49</td>
<td>348.826</td>
<td>348.93</td>
</tr>
<tr>
<td>50 years at 5%</td>
<td>1210.68</td>
<td>1214.45</td>
<td>1216.73</td>
<td>1217.49</td>
</tr>
</tbody>
</table>

The formula for computing this table is $A(t) = 100 \left(1 + \frac{5}{100 \times k}\right)^{kt}$ with $t = 25$ or $t = 50$.

In each of the two rows the amount of money available increases as the number of compoundings per year increases, but not by a lot. Over a 50 year period the difference between compounding 10 times or 100 times per year is a little less than $7 dollars.

Many financial institutions offer continuous compounding. By this they mean the value you would get if in the formula for compounding $k$ times per year you let $k$ become arbitrarily large. That is, if you invest $P$ dollars at $100r\%$ compounded continuously, you have

$$\text{amount of money after } t \text{ years} = \lim_{k \to \infty} P \left(1 + \frac{r}{k}\right)^{kt}$$

$$= P \left(\lim_{k \to \infty} \left(1 + \frac{r}{k}\right)^{kt}\right)$$

$$= P e^{rt}$$

Note: in this formula the number $r$ is not the percent, but the percent divided by 100.

We restate the above formula to underscore its importance:

If $P$ dollars is invested for $t$ years at $100r\%$ **compounded continuously**, then after $t$ years the investment will be worth

$$Pe^{rt}.$$  

We saw in an earlier page why this last equation is true. By the way, this is one of several reasons that the number $e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^{n}$ is chosen as a base for the natural exponential function.

The table below repeats the first table with the addition of a last column for continuous compounding. The formula to calculate the last row is $A(t) = 100e^{0.05t}$ with $t = 25$ or $t = 50$.

<table>
<thead>
<tr>
<th></th>
<th>$k = 10$</th>
<th>$k = 20$</th>
<th>$k = 50$</th>
<th>$k = 100$</th>
<th>continuous compounding</th>
</tr>
</thead>
<tbody>
<tr>
<td>25 years at 5%</td>
<td>347.95</td>
<td>348.49</td>
<td>348.826</td>
<td>348.93</td>
<td>$100e^{0.05 \times 25} \approx 349.03$</td>
</tr>
<tr>
<td>50 years at 5%</td>
<td>1210.68</td>
<td>1214.45</td>
<td>1216.73</td>
<td>1217.49</td>
<td>$100e^{0.05 \times 50} \approx 1218.25$</td>
</tr>
</tbody>
</table>

Once again we notice that there is not that much difference in the amount earned over 50 years if we have compounding occurring continuously or not. However, we should bear in mind that if we are the financial institute with millions of dollars we’re paying interest on, the difference between compounding monthly, or quarterly compared to continuously compounding can amount to a large sum of money.
Example 10: Suppose you have $1000 to invest at 8% compounded continuously. How much money will you have after 15 years?

Solution: Since the investment is earning interest which is compounded continuously, we use the formula $A(t) = Pe^{rt}$. In this example $P = 1000$, $r = 0.08$, and $t = 15$.

\[
A(t) = 1000e^{0.08 \times 15} \\
\approx 1000(3.3201169) \\
\approx 3320.12.
\]

So after 15 years the investment will have more than tripled.

Example 11: Which earns more money: to invest at 6% compounded quarterly for 10 years, or at 5.5% compounded continuously for 10 years?

Solution: Let’s suppose we have $P$ dollars to invest. The amount of money after 10 years in each investment will be

- 6% compounded quarterly: $P \left(1 + \frac{0.06}{4}\right)^{40} \approx P \times 1.814$
- 5.5% compounded continuously: $Pe^{0.055 \times 10} \approx P \times 1.733$

It is clear that the investment which earns 6% compounded quarterly does better than one earning 5.5% compounded continuously. Notice too, that the amount invested, $P$ dollars, has no effect on which investment is better.

Example 12: How much money will an investment of $1000 be worth if it is invested at 9.75% compounded quarterly for 6 years?

Solution: The formula which is applicable here is:

\[
A(t) = P \left(1 + \frac{r}{100 + kr}\right)^{kt},
\]

where $P = 1000$, $r = 9.75$, $k = 4$, and $t = 6$.

\[
A(6) = 1000 \left(1 + \frac{9.75}{400}\right)^{24} \\
\approx 1000(1.78244) \\
= 1782.44
\]

Example 13: How much money will an investment of $1000 be worth if it is invested at 9.75% compounded continuously for 6 years?

Solution: The appropriate formula to use here is $A(t) = Pe^{rt}$, where $P = 1000$, $r = 0.0975$, and $t = 6$.

\[
A(6) = 1000e^{0.0975 \times 6} \\
\approx 1000(1.794991) \\
\approx 1794.99.
\]
Example 14: How much money must be invested at 6.5% compounded continuously in order to have $1500 after 5 years?

Solution: The formula which relates the original investment, $P$, to what it grows to after 5 years is 
$$A(5) = P e^{0.065 \cdot 5}.$$ 
Since we want $A(5) = 1500$, this leads to the equation 
$$1500 = P e^{0.065 \cdot 5} \approx P(1.3840306)$$ 
Solving this for $P$ we have 
$$P \approx \frac{1500}{1.3840306} \approx 1083.79.$$ 
Thus, if you invest $1083.69 at 6.5% for 5 years the original investment will have grown to $1500.

Example 15: Which of the following investments earns the most money over a 15 year period? 6% compounded monthly, 5.875% compounded quarterly, or 5.9% compounded continuously

Solution: Before doing any computing we notice that the interest rate for the investment which is compounded continuously is larger than the interest rate for the investment which is compounded quarterly. This, eliminates the investment which is compounded quarterly from contention because given two investment possibilities if one of them has not only a larger interest rate, but is also compounded more often, then it is a better investment.

To solve this problem we now have to compare whether 6% compounded monthly does better than 5.9% compounded continuously.

$$6\% \text{ compounded monthly} = P \times \left(1 + \frac{6}{1200}\right)^{12 \cdot 15} \approx P \times 2.4540936$$

$$5.9\% \text{ compounded continuously} = Pe^{0.059 \cdot 15} \approx P \times 2.4229844$$

Comparing the two factors of $P$ we see that the factor associated with the 6% growth is larger than the factor associated with continuous compounding. So we are better off investing our money at 6% even though it is only compounded monthly.