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Combining Functions

Sum, Difference, Product and Quotient

We can perform algebraic operations on functions much as we do on numbers.

For example, if \( f(x) = x^2 + 1 \) and \( g(x) = x - 4 \) then \( f(x) + g(x) = (x^2 + 1) + (x - 4) = x^2 + x - 3 \).

This new function is called the sum of \( f \) and \( g \), denoted by \( f + g \). The difference, product, and quotient of two functions are defined in a similar manner.

If \( f \) and \( g \) are functions with domains \( A \) and \( B \) respectively, we define the following operations on functions:

1. **SUM:** \( (f + g)(x) = f(x) + g(x) \) \quad Domain: \( A \cap B \)
2. **DIFFERENCE:** \( (f - g)(x) = f(x) - g(x) \) \quad Domain: \( A \cap B \)
3. **PRODUCT:** \( (fg)(x) = f(x) \cdot g(x) \) \quad Domain: \( A \cap B \)
4. **QUOTIENT:** \( \left( \frac{f}{g} \right)(x) = \frac{f(x)}{g(x)} \) \quad \( g(x) \neq 0 \) \quad Domain: \( A \cap B - \{ x \mid g(x) = 0 \} \)

**Example 1:** Let \( f(x) = 2x^2 + x + 5 \) and \( g(x) = 3x^2 + 2x - 1 \), find the following:

a) Domain of \( f = (\infty, +\infty) \)

b) Domain of \( g = (\infty, +\infty) \)

c) \( (f + g)(x) = f(x) + g(x) = (2x^2 + x + 5) + (3x^2 + 2x - 1) = 5x^2 + 3x + 4 \)
d) Domain of \( f + g = (-\infty, +\infty) \)

e) \( (f - g)(x) = f(x) - g(x) = (2x^2 + x + 5) - (3x^2 + 2x - 1) = -x^2 - x + 6 \)
f) Domain of \( f - g = (-\infty, +\infty) \)

g) \( (fg)(x) = f(x) \cdot g(x) = (2x^2 + x + 5) \cdot (3x^2 + 2x - 1) = 6x^4 + 7x^3 + 15x^2 + 9x - 5 \)
h) Domain of \( fg = (-\infty, +\infty) \)

i) \( \left( \frac{f}{g} \right)(x) = \frac{f(x)}{g(x)} = \frac{(2x^2 + x + 5)}{(3x^2 + 2x - 1)} = \frac{(2x^2 + x + 5)}{(3x - 1)(x + 1)}, \quad x \neq \frac{1}{3}, -1 \)
j) Domain of \( \frac{f}{g} = (-\infty, +\infty) - \{ \frac{1}{3}, -1 \} = (-\infty, -1) \cup \left( -1, \frac{1}{3} \right) \cup \left( \frac{1}{3}, +\infty \right) \)

(That is, all real numbers except those that cause the denominator to be 0, namely \( \frac{1}{3} \) and \(-1\))
Example 2: Let \( f(x) = \sqrt{x-3} \) and \( g(x) = x - 5 \), find \( f + g \), \( f - g \), \( fg \), \( \frac{f}{g} \), and their domains.

Solution:
- Domain \([f] = [3, +\infty)\) \quad Domain \([g] = (-\infty, +\infty)\)
  \( x - 3 \geq 0 \) \quad No fractions or radicals.
  \( x \geq 3 \)
- \((f + g)(x) = \sqrt{x-3} + (x - 5)
  \) Domain \([f + g] = [3, +\infty) \cap (-\infty, +\infty) = [3, +\infty)\)
- \((f - g)(x) = \sqrt{x-3} - (x - 5) = \sqrt{x-3} - x + 5
  \) Domain \([f - g] = [3, +\infty)\)
- \((fg)(x) = (\sqrt{x-3})(x - 5) = x\sqrt{x-3} - 5\sqrt{x-3}
  \) Domain \([fg] = [3, +\infty)\)
- \(\left(\frac{f}{g}\right)(x) = \frac{\sqrt{x-3}}{x-5}, \quad x \neq 5
  \) Domain \(\left[\frac{f}{g}\right] = [3, 5) \cup (5, +\infty)\)
  Since \(g(5) = 0\), we must omit 5 from the domain of \(\frac{f}{g}\).

Example 3: Let \( f(x) = \sqrt{1 + x} \) and \( g(x) = \sqrt{1 - x} \), find \( f + g \), \( f - g \), \( fg \), \( \frac{f}{g} \), and their domains.

Solution:
- Domain \([f] = [-1, +\infty)\) \quad Domain \([g] = (-\infty, 1]\)
  \( 1 + x \geq 0 \) \quad \( 1 - x \geq 0 \)
  \( x \geq -1 \) \quad \( -x \geq -1 \)
  \( x \leq 1 \)
- \((f + g)(x) = \sqrt{1 + x} + \sqrt{1 - x} \quad\)
  Domain \([f + g] = [-1, +\infty) \cap (-\infty, 1] = [-1, 1]\)
- \((f - g)(x) = \sqrt{1 + x} - \sqrt{1 - x} \quad\)
  Domain \([f - g] = [-1, 1]\)
- \((fg)(x) = (\sqrt{1 + x})(\sqrt{1 - x}) = \sqrt{(1 + x)(1 - x)} = \sqrt{1 - x^2}
  \) Domain \([fg] = [-1, 1]\)
- \(\left(\frac{f}{g}\right)(x) = \frac{\sqrt{1 + x}}{\sqrt{1 - x}}, \quad x \neq 1
  \) Domain \(\left[\frac{f}{g}\right] = [-1, 1) - \{1\} = [-1, 1)\)
  Since \(g(1) = 0\), we must omit 1 from the domain of \(\frac{f}{g}\).
Sums, differences, products, and quotients of functions are defined pointwise. For example, the sum \((f + g)(x)\) is found by evaluating \(f(x)\) and \(g(x)\) and adding the resulting functional values.

**Example 4:** Find the pointwise sum of the graphs shown below.

![Graph of functions](image)

a) Use the graphs of \(f\) and \(g\) to complete the table of values.

<table>
<thead>
<tr>
<th>(x)</th>
<th>(f(x))</th>
<th>(g(x))</th>
<th>((f + g)(x))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(-5)</td>
<td>2</td>
<td>-2</td>
<td>0</td>
</tr>
<tr>
<td>(-4)</td>
<td>0</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>(-3)</td>
<td>-2</td>
<td>0</td>
<td>-2</td>
</tr>
<tr>
<td>(-2)</td>
<td>-4</td>
<td>1</td>
<td>-3</td>
</tr>
<tr>
<td>(-1)</td>
<td>-3</td>
<td>2</td>
<td>-1</td>
</tr>
<tr>
<td>0</td>
<td>-2</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>-1</td>
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<td>1</td>
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<tr>
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<td>1</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>-2</td>
<td>2</td>
</tr>
<tr>
<td>5</td>
<td>2</td>
<td>-4</td>
<td>-2</td>
</tr>
</tbody>
</table>

b) Use the points to draw the graph of \(f + g\).

Solution:

a) Plot the points \((x, (f + g)(x))\) from your table and draw the lines to get the graph of \(f + g\).
Example 5: Use graph paper to graph the functions \( f(x) = x^2 + 1 \) and \( g(x) = x - 4 \) on the same axes.

a) Use point-wise addition to find the graph of their sum.

b) Determine the rule for \( f + g \) algebraically.

c) What shape should the graph of the \( f + g \) have?

Solution:

\[ \text{red: } f(x) = x^2 + 1 \quad \text{blue: } g(x) = x - 4 \]

a) \text{red: } x^2 + 1 \quad \text{blue: } x - 4 \quad \text{maroon: } f + g

b) \( (f + g)(x) = f(x) + g(x) = (x^2 + 1) + (x - 4) = x^2 + x - 3. \)

c) A parabola.