Complex Numbers

Geometry of Complex Arithmetic

In the next few pages we will discuss geometrical interpretations of addition and multiplication of complex numbers. If you are not familiar with the Cartesian coordinate system for the plane, defer reading this material until you have learned how to coordinatize the plane.

We first want to associate with every complex number a unique point in the $x,y$ plane. We do this in the following manner:

If $z = a + bi$ is a complex number in standard form, then we associate the ordered pair of real numbers $(a,b)$ with $z$. The point in the plane which is associated with $z = a + bi$ is that point with Cartesian coordinates $(a,b)$.

If $a$ is a real number, then $a$, written as a complex number, has the form $a + 0i$. This means that the point in the plane we associate with $a$ has coordinates $(a,0)$. Thus, real numbers are associated with the points on the $x$-axis.

The figure below plots the points $1 - 2i, 3 + i, -3 - 3i, \text{ and } -2$.

![Diagram showing points on the plane](attachment:image.png)

To add two complex numbers geometrically one constructs the diagonal of the parallelogram determined by the two numbers being added.

That is, if $z_1 = a_1 + b_1i$ and $z_2 = a_2 + b_2i$ are added together, then the points in the plane which correspond to the three complex numbers $z_1, z_2, \text{ and } z_1 + z_2$ have coordinates:

$(a_1, b_1), (a_2, b_2), \text{ and } (a_1 + b_1, a_2 + b_2)$ respectively.

A picture of this is shown below.

![Diagram showing addition of complex numbers](attachment:image.png)
The sides of the parallelogram (the dashed arrows) are the lines from the origin to the points \((a_1, b_1)\), and \((a_2, b_2)\) respectively. The diagonal (the solid arrow) of the parallelogram is the line from the origin to the point with coordinates \((a_1 + b_1, a_2 + b_2)\).

To visualize subtraction is easy, just add the negative of the number being subtracted. That is, \(z_1 - z_2 = z_1 + (-z_2)\). To construct \(z_1 - z_2\) geometrically, draw the parallelogram whose sides are determined by the points \((a_1, b_1)\) and \((-a_2, -b_2)\).

For obvious reasons adding complex numbers is sometimes referred to as the parallelogram law of addition.

**Example 1**: Sketch the sum of \(1 - i\) and \(3 + i\).

**Solution**: Let’s add these numbers algebraically first.

\[
(1 - i) + (3 + i) = 4.
\]

Figure (a) below shows the complex numbers \(1 - i\) and \(3 + i\). Figure (b) shows the parallelogram they generate and the diagonal of the parallelogram.

**Example 2**: Sketch the difference of the complex numbers \(1 + 2i\) and \(3 - i\).

**Solution**: Let’s subtract them algebraically first.

\[
(1 + 2i) - (3 - i) = (1 - 3) + (2 + 1)i = -2 + 3i.
\]

Figure (a) below shows a sketch of the complex numbers \(1 + 2i\) and \(-(3 - i) = -3 + i\). Figure (b) shows the parallelogram they generate and the diagonal of the parallelogram which is the difference of \(1 + 2i\) and \(3 - i\).
Our next project is to give a geometric description of multiplication of complex numbers.

First let’s look at the geometry of multiplication by \(i\). In Figure (a) below we multiply 1 by \(i\), and in Figure (b) we multiply \(1 + i\) by \(i\). In both cases the result of multiplication by \(i\) is a 90 degree counterclockwise rotation.

If we multiply the complex number \(a + bi\) by \(i\), the result is \(i(a + bi) = -b + ai\). It can be shown that the lines from the origin to the points \((a, b)\) and \((-b, a)\) are perpendicular. This means that multiplication by \(i\) corresponds to a 90 degree rotation.

It can be shown in more advanced courses, that if \(z_1 = a_1 + b_1i\) and \(z_2 = a_2 + b_2i\) are two complex numbers, and \(\theta_1\) and \(\theta_2\) are the angles between the positive x-axis and the line segments going from the origin to the points \((a_1, b_1)\) and \((a_2, b_2)\) respectively, then the angle made by the product \(z_1z_2\) is \(\theta_1 + \theta_2\). This is illustrated in the figure below.