Exponential Functions

Review of Exponents

In this chapter we will define and discuss the properties of exponential functions. These are functions of the form \( a^x \) where \( a \) is a positive real number, and \( x \) is any real number. Before discussing this function, we’ll quickly review the laws of exponents, and then show how \( a^x \) is defined for irrational numbers. A reminder of terminology: \( a \) is called the base and \( x \) is called the power or exponent.

Below we quickly review exponentiation, and the laws of exponents. For a more thorough review see the previous chapter on Exponents. The following lists those \( x \) for which we can compute \( a^x \), and how to do so. Remember \( a \) is any positive real number.

1. If \( x = 0 \), then \( a^x \) is defined to be 1.
2. If \( x \) is a positive integer, then \( a^x \) is computed by multiplying \( a \) by itself \( x \) times. Thus, \( a^5 = aaaaa \).
3. If \( x \) is a negative integer, then \( a^x = \left( \frac{1}{a} \right)^{-x} \). Thus, \( a^{-5} = \left( \frac{1}{a} \right)^{5} \).
4. If \( x \) is the reciprocal of an integer, e.g., \( x = \frac{1}{5} \), then \( a^x = b \) where \( b^{1/x} = a \). Note: if \( x = 1/5 \), then \( 1/5 = 5 \). Thus, \( a^{1/5} = b \), if and only if \( b^5 = a \).
   That is, \( a^x \) is the \( x \)th root of \( a \). There is of course a computational problem here. It may not be easy to compute the \( x \)th root of \( a \).
5. If \( x \) is a rational number, i.e., \( x = \frac{m}{n} \) where \( m \) and \( n \neq 0 \) are integers, then \( a^x = (a^m)^{1/n} \).

The table below lists the algebraic properties of exponents:

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( a^m \cdot a^n = a^{m+n} )</td>
</tr>
<tr>
<td>2</td>
<td>( \frac{a^m}{a^n} = a^{m-n} )</td>
</tr>
<tr>
<td>3</td>
<td>( a^{-n} = \frac{1}{a^n} )</td>
</tr>
<tr>
<td>4</td>
<td>( a^0 = 1 ) for ( a \neq 0 )</td>
</tr>
<tr>
<td>5</td>
<td>( (ab)^n = a^nb^n )</td>
</tr>
<tr>
<td>6</td>
<td>( \left( \frac{a}{b} \right)^n = \frac{a^n}{b^n} )</td>
</tr>
<tr>
<td>7</td>
<td>( (a^m)^n = a^{mn} )</td>
</tr>
</tbody>
</table>

If you don’t already have these rules memorized, stop right now, and memorize them.

**Question:** What does \( (16)^{-3/4} \) equal?

**Answer:**

\[
(16)^{-3/4} = (2^4)^{-3/4} = (2)^{-3} = 8^{-1} = \frac{1}{8}
\]

The problem we now face is what does \( a^x \) mean if \( x \) is an irrational number? For example what do \( 2^\pi \), or \( 3^{\sqrt{2}} \) equal? See the next page for a discussion of this.
In this page the computation of \( a^x \) is discussed for the case when \( x \) is irrational. We need one property of rational numbers before we can compute \( a^x \), and this property is:

Given any irrational number \( x \), there is a rational number \( \frac{m}{n} \) which is as close to \( x \) as we want.

Another way to express this, is to say that any irrational number can be approximated as closely as we desire with a rational number.

This is the key to calculating (approximating) \( a^x \), we find a rational number \( \frac{m}{n} \) which is very close to \( x \), and then compute \( a^{\frac{m}{n}} \). This number, \( a^{\frac{m}{n}} \), can then be shown to be close to something. This something is called \( a^x \).

The plot below shows \( 2^x \) for various rational values of \( x \) between \(-1\) and \(1\), they are the red circles. The blue curve is the plot of \( 2^x \) on the interval \(-1\) to \(1\).

![Plot of 2^x](image)

There are only a few red dots and lots of blue. This is the general situation. There are a lot more irrational numbers then there are rational numbers, but the amazing thing is that any irrational number can be approximated with a rational number.

In the following example we use a calculator to compute \( 2^x \) for a sequence of rational numbers \( x \) which are getting close to the irrational number \( \sqrt{2} \). These numbers \( 2^x \) will be getting close to \( 2^{\sqrt{2}} \).

**Example 1:** Compute \( 2^x \) for a sequence of \( x \)'s getting close to \( \sqrt{2} \).

Solution: The rational numbers we will use to approximate \( \sqrt{2} \) are 1.4, 1.41, 1.414, 1.4142, 1.41421, and finally 1.414213. The table below list these values of \( x \) and below them the corresponding values of \( 2^x \).

<table>
<thead>
<tr>
<th>( x )</th>
<th>1.4</th>
<th>1.41</th>
<th>1.414</th>
<th>1.4142</th>
<th>1.41421</th>
<th>1.414213</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 2^x )</td>
<td>2.63902</td>
<td>2.65737</td>
<td>2.66475</td>
<td>2.66512</td>
<td>2.66514</td>
<td>2.66514</td>
</tr>
</tbody>
</table>

The value of \( 2^{\sqrt{2}} \) as computed by our calculator to 7 decimal places is

\[
2^{\sqrt{2}} = 2.6651441
\]

Just imagine how long it would have taken to compute \( 2^{1.414} \) without the aid of our calculator, and even then we are only within 2 decimal place accuracy of \( 2^{\sqrt{2}} \).

We list one more time the laws of exponents. This time with the remark that the powers are now allowed to be any real number.

For any positive real number \( a \), and any real numbers \( x \) and \( y \), the following properties hold:

1. \( a^x a^y = a^{x+y} \)
2. \( \frac{a^x}{a^y} = a^{x-y} \)
3. \( a^{-x} = \frac{1}{a^x} \)
4. \( a^0 = 1 \) for \( a \neq 0 \)
5. \( (ab)^x = a^x b^x \)
6. \( \left( \frac{a}{b} \right)^x = \frac{a^x}{b^x} \)
7. \( (a^x)^y = a^{xy} \)