Introduction to Functions

Definition of Function

In almost every aspect of our lives, we find examples of situations where one quantity depends on another. For example, the wind chill depends on the speed of the wind; the area of a circle depends on its radius; and the amount earned by your investment depends on the interest rate.

Definitions:
- A function is a relation in which no two different ordered pairs have the same first element.
  \[ F = \{(2,3), (5,7), (4,-2), (9,1), (8,4), (6,3)\} \]
- A function can also be defined as a rule that assigns exactly one element in set B to each element in set A.
- Set A, the set of all first coordinates, is called the domain of the function.
- Set B, the set of all second coordinates, is called the range.

In the function \( F \) above, the domain is the set of first elements: \{2, 5, 4, 9, 8, 6\} and the range is the set of second elements: \{3, 7, -2, 1, 4\}.

The function \( F \) is depicted pictorially below:

The characteristic that distinguishes a function from any other relation is that there is only one output (\( y \)-value) for each input (\( x \)-value).

Example 1: Which of the following relations are functions?

a) \{\( (2,3), (-1,3), (5,3), (8,3), (0,3) \} \quad \text{b) } \{\begin{array}{c} -2 \\ -1 \\ 0 \\ 1 \end{array} \begin{array}{c} 3 \\ 2 \\ 1 \\ 0 \end{array} \}

b) \{\begin{array}{c} -2 \\ -1 \\ 0 \\ 1 \end{array} \begin{array}{c} 3 \\ 2 \\ 1 \\ 0 \end{array} \}

c) \{\begin{array}{c} 1,5 \end{array} \begin{array}{c} 2,7 \end{array} \begin{array}{c} 3,9 \end{array} \begin{array}{c} 1,4 \end{array} \}

Solution:
- a) Function. Although each output is the same, 3, there is only one output for each input so the relation is a function.
- b) Function. Each \( x \) is paired with only one \( y \).
- c) Not a function. 1 is paired with two outputs: 5 and 4. We have two ordered pairs with the same first element.
Example 2: Many functions are programmed into your calculator. Use the $\sqrt{}$ function to find the outputs for the following inputs.

<table>
<thead>
<tr>
<th>Input</th>
<th>$\sqrt{}$</th>
<th>Output</th>
</tr>
</thead>
<tbody>
<tr>
<td>25</td>
<td>$\sqrt{25}$</td>
<td>5</td>
</tr>
<tr>
<td>9</td>
<td>$\sqrt{9}$</td>
<td>3</td>
</tr>
<tr>
<td>0</td>
<td>$\sqrt{0}$</td>
<td>0</td>
</tr>
<tr>
<td>$\frac{1}{4}$</td>
<td>$\sqrt{\frac{1}{4}}$</td>
<td>$\frac{1}{2}$</td>
</tr>
<tr>
<td>$-4$</td>
<td>$\sqrt{-4}$</td>
<td>ERROR</td>
</tr>
</tbody>
</table>

a) What output did you get for $-4$?

b) List 4 numbers that are in the domain of $y = \sqrt{x}$ and 4 numbers in its range.

Solution:

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<tr>
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</tr>
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</tr>
<tr>
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a) ERROR on your calculator means that $\sqrt{-4}$ is not a real number. Therefore, $-4$ is not in the domain of the function.

b) Answers will vary. Domain will be numbers you key into your calculator (input). Range will be the output numbers that the calculator provides on your screen.

Example 3: Which of the following represent functions? Determine the domain and range of each set.

a) $\{(2, 5), (-1, 6), (5, 7), (8, 5), (2, 4)\}$

b) $\{(95, A), (82, B), (92, A), (62, D), (74, C), (85, B), (58, F), (77, C)\}$

c) $\begin{array}{c|c}
 x & y \\
-1 & 5 \\
0 & 3 \\
1 & -1 \\
2 & -3 \\
3 & -5 \\
\end{array}$

d) $\begin{array}{c|c|c|c|c}
 Input & 6 & 2 & 1 & 3 & 6 \\
 Output & 7 & 9 & 4 & 7 & 5 \\
\end{array}$

Solution:

a) Not a function. 2 is paired with both 5 and 4. Domain: $\{2, -1, 5, 8\}$ Range: $\{5, 6, 7, 4\}$

b) Function. Domain: $\{95, 82, 92, 62, 74, 85, 58, 77\}$ Range: $\{A, B, C, D, F\}$

c) Function. Domain: $\{-1, 0, 1, 2, 3\}$ Range: $\{5, 3, -1, -3, -5\}$

d) Not a function. 6 is paired with both 7 and 5. Domain: $\{6, 2, 1, 3\}$ Range: $\{7, 9, 4, 5\}$