Graphs of Equations

Symmetry

When we think of symmetry in art, we think of balance on both sides of a center of focus. In other situations, the term conjures up the idea of a “mirror image.” In either case, the notion of symmetry means essentially the same thing in an algebra class: a shape looks the same on both sides of a dividing line or point. The dividing line is called the line of symmetry. We will also consider points of symmetry in this section.

Examine the graph of the circle \( x^2 + y^2 = 25 \) for symmetry.

Question: How many lines or points of symmetry can you find for the circle?
Answer: The circle has symmetry about its center—a point, and about any straight line that goes through its center—an infinite number of lines.

Although any line that goes through the center of the circle can be a line of symmetry for the circle, we will focus our efforts in this section on the \( x \)- and \( y \)-axes. We will also look at symmetry about the origin.

Example 1: The figures on the axes below are symmetric about the \( x \)-axis. Find the coordinates of \( A, B, \) and \( C \).

Solution: \( A(1, -5) \quad B(4, -7) \quad C(3, -2) \)

Note that the \( x \)-coordinates are the same as the corresponding point in the Quadrant I figure, but the \( y \)-coordinates are the negatives of the \( y \)-coordinates in the Quadrant I figure.

Example 2: Find the points of the figure that would be symmetric, about the \( y \)-axis, to the above figure in Quadrant I.
Solution: Corresponding points would be \((-1, 5) \quad (-4, 7) \quad (-3, 2)\).

Note that the \( y \)-coordinates of corresponding points are the same as the Quadrant I figure, but the \( x \)-coordinates are the negatives.
Symmetric about the y-axis.

When we say that a graph is symmetric with respect to the y-axis we mean that if the graph is reflected through or reflected about the y-axis we get the same graph.

Note that the graph of the equation \( y = 2x^2 - 2 \) is symmetric about the y-axis. That is, if we reflected the graph 180° about the y-axis, the graph would be the same. See the plot below.

Examine the table of values that we used to graph \( y = 2x^2 - 2 \).

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>6</td>
</tr>
<tr>
<td>-1.5</td>
<td>2.5</td>
</tr>
<tr>
<td>-1</td>
<td>0</td>
</tr>
<tr>
<td>-0.5</td>
<td>-1.5</td>
</tr>
<tr>
<td>0</td>
<td>-2</td>
</tr>
<tr>
<td>0.5</td>
<td>-1.5</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>1.5</td>
<td>2.5</td>
</tr>
<tr>
<td>2</td>
<td>6</td>
</tr>
</tbody>
</table>

**Question:** What do you notice about the y-values when \( x = -2 \) and \( x = 2 \)? What about the y-values for \( x = -1.5 \) and \( x = 1.5 \)?

**Answer:** For any number \( x \) and its opposite \(-x\), the y-values are equal.

**Definition:** A graph is symmetric about the y-axis if and only if for every point \((x, y)\) on the graph, \((-x, y)\) will also be a point on the graph.

This definition leads us to the following test for symmetry about the y-axis.

**Test for Symmetry about the y-axis.**

To determine algebraically if a graph will be symmetric about the y-axis, substitute \(-x\) for \(x\) and simplify. If the resulting equation is equivalent to the original equation, the graph will be symmetric about the y-axis.

**Example 3:** Show that \( 4x^2 - 2y = 4 \) has symmetry with respect to the y-axis.

**Solution:** Substituting \(-x\) for \(x\) in the original equation \( 4x^2 - 2y = 4 \), we get \( 4(-x)^2 - 2y = 4 \).

Simplifying, \( 4x^2 - 2y = 4 \).

The result is equivalent to the original equation; therefore, the graph of \( 4x^2 - 2y = 4 \) is symmetric about the y-axis.

**Example 4:** Test the equation \( x^2y - 5y^3 = x^4 \) for symmetry about the y-axis.

**Solution:** Substitute \(-x\) for \(x\) and simplify

\[ (-x)^2y - 5y^3 = (-x)^4 \implies x^2y - 5y^3 = x^4 \]

Since the resulting equation is the same as the original, the graph is symmetric about the y-axis.
The next type of symmetry we discuss is that of a graph being symmetric with respect to the $x$-axis. The following graph has symmetry about the $x$-axis. That is, the graph will be the same if it is reflected $180^\circ$ about the $x$-axis. For this to occur, every $x$ must be paired with both $y$ and $-y$.

\[
(x, y) \\
(x, -y) \\
(x, y) \\
(x, -y)
\]

The graph will be the same if it is reflected $180^\circ$ about the $x$-axis.

**Definition:** A graph has symmetry about the $x$-axis if and only if $(x, -y)$ is a point on the graph for every point $(x, y)$ on the graph.

**Test for Symmetry about the $x$-axis:**
Substitute $-y$ for $y$ in the equation and simplify. If the resulting equation is equivalent to the original, the graph will be symmetric about the $x$-axis.

**Example 5:** Test $x^2y - 5y^3 = x^4$ for symmetry about the $x$-axis.
**Solution:** Substitute $-y$ for $y$ and simplify

\[
x^2(-y) - 5(-y)^3 = x^4 \implies -x^2y + 5y^3 = x^4
\]

Multiply both sides by $-1$:

\[
-1(-x^2y + 5y^3) = -1(x^4)
\]

\[
x^2y - 5y^3 = -x^4
\]

Since neither of the resulting equations is equivalent to the original, the graph of the equation is not symmetric about the $x$-axis.

The following graph has symmetry about the origin. That is, it will be the same if it is rotated $180^\circ$ about the origin. Note that for every point $(x, y)$, the point $(-x, -y)$ will also be on the graph. Furthermore, the line segment connecting these corresponding pairs will go through the origin.

\[
(-1, 3) \\
(x, y) \\
(-x, -y) \\
(1, -3)
\]

**Definition:** The graph of an equation will have symmetry about the origin if and only if for every point $(x, y)$ on the graph, the point $(-x, -y)$ will also be on the graph.
Test for Symmetry about the origin:
Substitute \(-x\) for \(x\), and \(-y\) for \(y\) into the equation. If the resulting equation is equivalent to the original, the graph has symmetry about the origin.

Example 6: Test \(x^2y - 5y^3 = x^4\) for symmetry about the origin.
Solution: Substitute \(-x\) for \(x\), and \(-y\) for \(y\), and simplify.
\[ (-x)^2(-y) - 5(-y)^3 = (-x)^4 \Rightarrow -x^2y + 5y^3 = x^4 \]
Since the resulting equation is not equivalent to the original equation, there is no symmetry about the origin.

Note that even if we multiply the resulting equation by \(-1\), the result \(x^2y - 5y^3 = -x^4\) is not equivalent to the original equation. The results of the examples in this section have shown that \(x^2y - 5y^3 = x^4\) has symmetry about the \(y\)-axis only.

Example 7: Test the equation \(xy^3 - 5x^3y = xy\) for symmetry about the \(x\)-axis, \(y\)-axis, and origin.
Solution:
- \(y\)-axis: [Substitute \(-x\) for \(x\):]
  \[ (-x)(-y)^3 - 5(-x)^3(-y) = (-xy) \]
  \[ -xy^3 + 5x^3y = -xy \]
  \[ -xy^3 - 5x^3y = -xy \]
  \[ xy^3 - 5x^3y = xy \]
  Since the resulting equation is equivalent to the original, the graph has symmetry about the \(y\)-axis.
- \(x\)-axis: [Substitute \(-y\) for \(y\):]
  \[ x(-y)^3 - 5x^3(-y) = x(-y) \]
  \[ -xy^3 + 5x^3y = -xy \]
  \[ -(xy^3 - 5x^3y) = -(xy) \]
  \[ xy^3 - 5x^3y = xy \]
  \[ \Rightarrow \text{equation has symmetry about the } x\text{-axis}. \]
- origin: [Substitute \(-x\) for \(x\) and \(-y\) for \(y\):]
  \[ (-x)(-y)^3 - 5(-x)^3(-y) = (-x)(-y) \]
  \[ xy^3 - 5x^3y = xy \]
  \[ \Rightarrow \text{equation has symmetry about the origin}. \]
Example 8: Test the equation \( y = x|x^4 - 3| \) for symmetry about the \( x \)-axis, \( y \)-axis, and origin.

Solution:
- **y-axis:** [Substitute \(-x \) for \( x \):]
  \[
y = (-x)|(-x)^4 - 3| \quad \Rightarrow \text{equation does not have symmetry about the } y \text{-axis.}
\]
- **x-axis:** [Substitute \(-y \) for \( y \):]
  \[
  (-y) = x|x^4 - 3| \\
  -y = x|x^4 - 3| \Rightarrow \text{equation does not have symmetry about the } x \text{-axis.}
\]
- **origin:** [Substitute \(-x \) for \( x \) and \(-y \) for \( y \):]
  \[
  (-y) = (-x)|(-x)^4 - 3| \\
  -y = -x|x^4 - 3| \\
  y = x|x^4 - 3| \quad \Rightarrow \text{equation has symmetry about the origin.}
\]

Knowing whether the graph of a particular equation has symmetry about a point or line can assist us if we are graphing by plotting points. If we are using a grapher it provides us information about what our graph should look like so that we can determine whether a particular graph is reasonable for the equation we entered.

Consider the section of graph in the figure below.

If we knew that the graph had symmetry about the \( x \)-axis, \( y \)-axis, or origin, we could complete the graph knowing only what the Quadrant I section of the graph looked like.

Example 9: Reproduce the graph above using paper and pencil. Complete the graph for each of the following situations. The graph has symmetry about the

a) \( x \)-axis  
 b) \( y \)-axis  
 c) origin.

Solution:

(a)  

(b)  

(c)
To use symmetry in graphing an equation,
1. Determine whether the graph has symmetry about the x-axis, y-axis, or origin.
2. Plot enough points in the first quadrant to get an idea of what the graph looks like.
3. Apply symmetry to complete the graph.

Example 10: Use symmetry in graphing the equation \( xy = 2 \).

Solution: Determine symmetry.

- **x-axis:** \( x(-y) = 2 \)
  \(-xy = 2 \) \( \Rightarrow \) No symmetry about x-axis.

- **y-axis:** \( (-x)y = 2 \)
  \(-xy = 2 \) \( \Rightarrow \) No symmetry about y-axis.

- **origin:** \( (-x)(-y) = 2 \)
  \( xy = 2 \) \( \Rightarrow \) Symmetry about the origin.

Because \( xy = 2 \) has symmetry about the origin, use values of \( x \) that are positive in the table of values, and use symmetry to find the remainder of the graph.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/2</td>
<td>4</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>1/2</td>
</tr>
</tbody>
</table>

Since, for example, the point \((1/2, 4)\) is on the graph of \( xy = 2 \), and this graph is symmetric about the origin, we know that the point \((-1/2, -4)\) is also a point on the graph of \( xy = 2 \).