Linear Equations and Inequalities in Two Variables

Linear Equations in Two Variables

In this section we will examine equations in $x$ and $y$ where both variables are raised to the first power.

**Definition:** Any equation that can be written in the form $Ax + By = C$, where $A$ and $B$ are not both 0, is called a **linear equation**.

**Question:** According to the definition, which of the following are linear equations?

- a) $2x + y = 3$
- b) $x - 3 = 0$
- c) $y = -3x + 1$
- d) $x^2 + y = 1$
- e) $2x + y^2 = 5$

**Answer:**

- a) Yes. $A = 2, B = 1, C = 3$. $x$ and $y$ are raised to the first power.
- b) Yes. $x - 3 = 0$ can be written as $x + 0y = 3$, so that $A = 1, B = 0, C = 3$. $A$ and $B$ are not both 0, the variables are raised to the first power.
- c) Yes. $y = -3x + 1$ can be written as $3x + y = 1$. $A = 3, B = 1, C = 1$. Both variables are raised to the first power.
- d) No. $x$ is raised to the second power.
- e) No. $y$ is raised to the second power.

Linear equations can be graphed as any other equation by plotting points.

**Example 1:** Graph the linear equation $2x + y = 3$.

**Solution:** Solve for $y$.

\[ y = -2x + 3 \]

Fill in a table of values

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y$</th>
<th>$(x,y)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>7</td>
<td>(-2,7)</td>
</tr>
<tr>
<td>-1</td>
<td>5</td>
<td>(-1,5)</td>
</tr>
<tr>
<td>0</td>
<td>3</td>
<td>(0,3)</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>(1,1)</td>
</tr>
<tr>
<td>2</td>
<td>-1</td>
<td>(2,-1)</td>
</tr>
</tbody>
</table>

Plot the points and sketch.

Notice that all of the points were on a line. You might guess that this is why these equations are called linear equations.

The graph of any linear equation is a straight line. Because any two points determine a line, we can graph a linear equation by plotting at least two points. The intercepts are a good choice because they are usually easy points to find.

**Question:** What are the $x$- and $y$-intercepts of the above graph?

**Answer:**

- **y-intercept:** Substitute $x = 0$. $2(0) + y = 3 \implies y = 3$ or $(0,3)$
- **x-intercept:** Substitute $y = 0$. $2x + 0 = 3 \implies 2x = 3 \implies x = \frac{3}{2}$ or $\left(\frac{3}{2}, 0\right)$
Example 2: Graph the linear equation \(3x - 2y = -8\).
Solution: Solve for \(y\):
\[-2y = -3x - 8 \Rightarrow y = \frac{3}{2}x + 4\]
Make a table of values:
<table>
<thead>
<tr>
<th>(x)</th>
<th>(y)</th>
<th>((x, y))</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>1</td>
<td>(-2, 1)</td>
</tr>
<tr>
<td>0</td>
<td>4</td>
<td>(0, 4)</td>
</tr>
<tr>
<td>2</td>
<td>7</td>
<td>(2, 7)</td>
</tr>
<tr>
<td>4</td>
<td>10</td>
<td>(4, 10)</td>
</tr>
</tbody>
</table>

Plot the points and draw the line:
We can see from both the table and the graph that the \(y\)-intercept is 4.

Question: What is the \(x\)-intercept?
Answer: If \(y = 0\), \(3x - 2(0) = -8 \Rightarrow 3x = -8 \Rightarrow x = -\frac{8}{3}\). The \(x\)-intercept is \(-\frac{8}{3}\).

Example 3: Graph the linear equation \(y = -2x + 5\).
Solution: We recognize that \(y = -2x + 5\) is a linear equation.
Make a table of values:
<table>
<thead>
<tr>
<th>(x)</th>
<th>(y)</th>
<th>((x, y))</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>5</td>
<td>(0, 5)</td>
</tr>
<tr>
<td>-(\frac{5}{2})</td>
<td>0</td>
<td>((-\frac{5}{2}, 0))</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>(2, 1)</td>
</tr>
</tbody>
</table>

Plot the points and draw the line:

Example 4: Graph the linear equation \(2y + 6 = 0\).
Solution: We recognize that \(2y + 6 = 0\) is a linear equation. Solving for \(y\): \(y = -3\).
Note that the equation does not contain a variable \(x\). The only condition that exists for a point to be included as a solution to the equation is that \(y\) must be \(-3\).

Make a table of values:
<table>
<thead>
<tr>
<th>(x)</th>
<th>(y)</th>
<th>((x, y))</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>-3</td>
<td>(0, -3)</td>
</tr>
<tr>
<td>-5</td>
<td>-3</td>
<td>(-5, -3)</td>
</tr>
<tr>
<td>2</td>
<td>-3</td>
<td>(2, -3)</td>
</tr>
</tbody>
</table>

Plot the points and draw the line:
All points with \(y\)-coordinate \(-3\) lie on the horizontal line 3 units below the \(x\)-axis.
In the following investigation you may use a graphing calculator but record the table of values and the graph of each equation on graph paper before graphing the next equation. As you graph each equation, observe the results before graphing the next equation.

- If you have a TI-82 or TI-83, you can enter the equation using the "Y=" button on the top left of the display. (You must solve the equation for \( y \) first.)
- Set the viewing window using the "WINDOW" to the right of the "Y=" button.
- The following viewing window gives a FRIENDLY WINDOW: \( \text{xmin} = -4.7, \text{xmax} = 4.7, \text{xscl} \) tells you how many units you want each tick mark to represent. \( \text{ymin} = -3.1, \text{ymax} = 3.1 \) will give a SQUARE WINDOW.
- Otherwise, the range values can be manipulated to include the part of the graph you want to see. You can multiply each of the above settings to get other friendly and square windows.
- If you do not use a friendly window, you will get large decimals when you trace the graph using the "TRACE" button. Use right and left arrows to move the cursor over the graph. The "x=" and "y=" at the bottom of the screen will give the coordinates of the point highlighted by the cursor. The cursor may skip over the integral \( x \)-intercepts if the window is not a friendly window.

Example 5: Graph the following lines on the same coordinate system.

\[
\begin{align*}
y_1 &= x + 1 \\
y_2 &= 3x - 1 \\
y_3 &= \frac{1}{2}x + 1
\end{align*}
\]

a) Which number was different in each equation? What effect did this have on the graph?
b) Which number was the same in each equation? What did each line have in common?

Solution:

a) The coefficient of \( x \) was different in each equation. The coefficient of \( x \) affected the steepness of the graph.
b) The constant was 1 in each equation. They all had a \( y \)-intercept of 1.

Example 6: Graph the following lines on the same coordinate system.

\[
\begin{align*}
y_1 &= x \\
y_2 &= x + 2 \\
y_3 &= x - 1
\end{align*}
\]

a) Which number was different in each equation? What effect did this have on the graph?
b) What number was the same in each equation? What did each line have in common?

Solution:

a) The constant was different in each equation. The \( y \)-intercepts were different in each line.
b) The coefficient of \( x \) was 1 in each equation. Each line had the same steepness.

Example 7: Graph the following lines on the same coordinate system and compare with the lines in Example 5. What is different between the equation of the lines in this example and the lines in Example 5? What effect did this difference have on the graphs?

\[
\begin{align*}
y_1 &= -x + 1 \\
y_2 &= -3x + 1 \\
y_3 &= -\frac{1}{2}x + 1
\end{align*}
\]

Solution: The coefficients of \( x \) in Example 5 are positive, but they are negative in this example. Graphs in Example 5 go uphill from left to right. Graphs in this example go downhill from left to right.

Conclusions:

Graphs of lines in the form \( y = mx + b \).

- \( m \), the coefficient of \( x \), affects the steepness of the line. The larger \( |m| \), the steeper the line.
  - When \( m > 0 \), the line goes uphill from left to right. We say the line is increasing.
  - When \( m < 0 \), the line goes downhill from left to right. We say the line is decreasing.
- The number \( b \) is the \( y \)-intercept. If \( x = 0 \), then \( y = m(0) + b \Rightarrow y = b \)