Linear Equations and Inequalities in Two Variables

Slopes of Lines
The idea of steepness has applications in many areas of our lives. When we build a house, we must determine the steepness of a roof called the "pitch of the roof". Certainly if we are skiers or cyclists, we are interested in the steepness of a ski slope or roadway. In some cases it is important that we be able to measure steepness. For example, the Americans with Disabilities Act requires that a wheelchair ramp rise no more than 6 inches for 6 feet of horizontal distance, as shown below.

\[
\text{Ramp} \quad \{ \text{6 in.} = 0.5 \text{ ft.} \}
\]

In the above example, we would call the vertical change the rise and the horizontal distance the run. We describe the steepness as the ratio of the rise to the run.

Consider the line below that goes through the points \( A(2, 1) \) and \( B(6, 3) \).

The rise can be calculated by subtracting the \( y \)-coordinates of the given points and the run can be calculated by subtracting the \( x \)-coordinates of the points. The ratio of the rise to the run \( \frac{2}{4} \) can be simplified to \( \frac{1}{2} \). This ratio is called the slope of the line.

**Definition:** The slope of a line, \( m \), is the ratio of the change in \( y \) to the change in \( x \).

\[ m = \frac{\text{rise}}{\text{run}} = \frac{\text{change in } y \ (\Delta y)}{\text{change in } x \ (\Delta x)} \]

The following formula generalizes the process we used above to calculate the slope of any line if we are given two points on the line: \( P_1(x_1, y_1) \) and \( P_2(x_2, y_2) \). The rise \( \Delta y \) and the run \( \Delta x \) can be calculated as follows: \( \Delta y = y_2 - y_1 \) and \( \Delta x = x_2 - x_1 \).

**Definition:** The slope of the line through points \( P_1(x_1, y_1) \) and \( P_2(x_2, y_2) \) is

\[ m = \frac{y_2 - y_1}{x_2 - x_1} \]

**Example 1:** Plot each pair of points on graph paper and find the slope of the line:

- a) \((-3, 5)\) and \((4, -2)\)
- b) \((5, 7)\) and \((-1, 7)\)
- c) \((-2, -3)\) and \((5, 6)\)
- d) \((-2, 5)\) and \((-2, -1)\)

**Solution:**

- a) \( m = \frac{-2 - 5}{4 - (-3)} = \frac{-7}{7} = -1 \)
- b) \( m = \frac{7 - (-7)}{-1 - (-5)} = \frac{0}{-6} = 0 \)
- c) \( m = \frac{6 - (-3)}{5 - (-2)} = \frac{9}{7} \)
- d) \( m = \frac{-1 - 5}{-2 - (-2)} = \frac{-6}{0} = \text{undefined} \)
Example 2:  Use your results to complete each sentence.

a) The line goes uphill from left to right when the slope is ___.
b) The line goes downhill from left to right when the slope is __.
c) The line is horizontal when the slope is ____.
d) The line is vertical when the slope is ____.

Answers:  a) positive  b) negative  c) 0  d) undefined

Investigation of Slopes with a Grapher

Enter the equation \( y = 3x + 1 \) into your grapher. Use TABLE mode to answer the following questions.

Question:  How much does \( y \) change every time \( x \) increases by 1 unit?
Answer:  3 units. For example when \( x \) changes from 0 to 1, \( y \) changes from 1 to 4.

Question:  How much does \( y \) change every time \( x \) increases by 2 units?
Answer:  \( y \) changes 6 units. For example, when \( x \) changes from 0 to 2, \( y \) changes from 1 to 7, a total of 6 units.

Question:  How much does \( y \) change every time \( x \) increases by 3 units?
Answer:  9 units. For example, when \( x \) changes from 2 to 5, \( y \) changes from 7 to 16, a total of 9 units.

Question:  Write each of the above answers as a ratio: \( \frac{\text{change in } y}{\text{change in } x} = \frac{\Delta y}{\Delta x} \). Are these ratios equivalent?
Answer:  \( \frac{3}{1} = \frac{6}{2} = \frac{9}{3} \). All these ratios are equivalent to \( \frac{3}{1} \).

Question:  Compare the slope of the line and the coefficient of \( x \) in the equation of the line.
Answer:  The slope of the line is \( \frac{3}{1} = 3 \). The coefficient of \( x \) in the equation of the line is 3.

Example 3:  Enter the equation \( y = -\frac{2}{3}x + 4 \) into your grapher.

a) Use two points from your table to calculate the slope of the line. Compare with the coefficient of \( x \) in the equation.
b) Try several more examples of lines in the form \( y = mx + b \). What can you conclude about the relationship between the coefficient of \( x \) of a linear equation that has been solved for \( y \) and the slope of the line?

Solution:

a) \( m = -\frac{2}{3} \) which is also the coefficient of \( x \) in the equation.
b) The coefficient of \( x \) is the slope of the line.

Conclusions:

- The coefficient of \( x \) is the slope of the line.
- If the slope is positive, the graph goes uphill from left to right. We say the line is increasing.
- If the slope is negative, the graph goes downhill from left to right. We say the line is decreasing.

Not only can we find the slope of a line by solving its equation for \( y \), but the form \( y = mx + b \) is especially
handy in graphing lines. We must use the "y =" form to enter the equation into the grapher, but it also makes pencil and paper graphing easier, as well.

**To graph a line using the y = mx + b form,**
- Since the y-intercept is b, plot the point (0, b) on the y-axis.
- From the y-intercept, apply the slope, \( m = \frac{\text{change in } y}{\text{change in } x} = \frac{\Delta y}{\Delta x} = \frac{\text{rise}}{\text{run}} \) to locate a second point on the line.
- Draw the line.

**Example 4:** Graph \( y = -3x - 2 \).
Solution: The y-intercept is -2, so we plot the point (0, -2). The slope is \( -3 = \frac{-3}{1} \) which means that \( y \) decreases 3 units when \( x \) increases 1 unit. We then count one unit to the right and 3 units down from the y-intercept to plot our second point, and draw the line.

\[
\begin{align*}
\text{m} &= -3 \\
\text{rise} &= -3 \\
\text{run} &= 1
\end{align*}
\]

Since the slope of the line is negative, we expect the line to go downhill from left to right. We say this line is decreasing.

**Example 5:** Find the slope and y-intercept of the line \( 4x - 2y = 6 \). Then graph.
Solution: We must first solve the equation for \( y \):
\[
4x - 2y = 6 \
\Rightarrow -2y = -4x + 6 \
\Rightarrow y = 2x - 3
\]
Therefore, the slope \( m = 2 \) and the y-intercept \( b = -3 \). We first plot the point \( (0, -3) \). From that point we use the slope \( \frac{2}{1} \) to rise 2 and run 1. Plot the second point and draw the line.

We expect the line to go uphill from left to right since the slope is positive. We say this line is increasing.

**Horizontal and Vertical Lines**
Consider the horizontal line below:

Using the points to calculate the slope gives \( m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5 - 5}{2 - 3} = \frac{0}{-1} = 0. \)

Because the \( y \)-coordinates are equal on any horizontal line, \( y_2 - y_1 = 0 \), the slope of a horizontal line is 0.

In a similar manner we can determine the slope of any vertical line.

Calculating its slope gives \( m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4 - (-2)}{3 - 3} = \frac{6}{0} = \text{undefined}. \)

Because the \( x \)-coordinates are equal on any vertical line, \( x_2 - x_1 = 0 \).

The slope of a vertical line is undefined, or we say that a vertical line has no slope.

NOTE: The slope of a vertical line is NOT 0. Consider skiing along a horizontal line as compared to skiing down a vertical line. There is a HUGE difference!!

### Parallel and Perpendicular Lines

Relationships between pairs of parallel and perpendicular lines and their slopes exist and prove useful in a variety of mathematical settings.

- **Parallel Lines:** Equal slopes
- **Perpendicular Lines:** Slopes are negative reciprocals

**Example 6:** Investigate the following pairs of lines. If you use a calculator, use a square viewing window.
Draw the graphs of each pair of lines on a single set of axes and record the slope of each line in the pair.

\[
a) \quad y = \frac{1}{2}x - 5 \quad b) \quad y = -3x + 4 \quad c) \quad y = \frac{2}{3}x + 5 \quad d) \quad y = -4x + 2
\]

\[
\begin{align*}
y &= \frac{1}{2}x + 4 \quad & y &= -3x + 9 \quad & y &= -\frac{2}{3}x + 1 \quad & y &= \frac{1}{4}x
\end{align*}
\]

**Question:** What is your conclusion about lines whose slopes are equal?
**Answer:** Lines whose slopes are equal appear to be parallel.

**Question:** What is your conclusion about lines that are perpendicular?
**Answer:** Lines that have slopes that are negative reciprocals appear to be perpendicular.

**Example 7:** Find the slope of any line that is a) parallel to \((\parallel)\) and b) perpendicular to \((\perp)\) the following:
1. the line through the points \(A(3, -2)\) and \(B(-5, 8)\).
2. the line \(3x - 5y = 10\).

**Solution:**
1. \[m_{AB} = \frac{8 - (-2)}{-5 - 3} = -\frac{5}{4}\]
   a) slope of line parallel to \(AB\): \[m_1 = -\frac{5}{4}\] (same slope)
   b) slope of line perpendicular to \(AB\): \[m_\perp = \frac{4}{5}\] (negative reciprocal)
2. Slope of line \(3x - 5y = 10\):
   \[-5y = -3x + 1 \quad \Rightarrow \quad y = \frac{3}{5}x - \frac{1}{5} \quad \Rightarrow \quad m = \frac{3}{5}\]
   a) \[m_1 = \frac{3}{5}\]
   b) \[m_\perp = -\frac{5}{3}\]

**Example 8:** If the line through \((3, 6)\) and \((2, y_2)\) is parallel to \(3x - 4y = 8\), find the value for \(y_2\).

**Solution:** Find the slope of the given line by solving for \(y:\)
\[-4y = -3x + 8 \quad \Rightarrow \quad y = \frac{3}{4}x - 2. \] The slope of the given line is \(m = \frac{3}{4}\).

Since parallel lines have equal slopes, the slope of the line through \((3, 6)\) and \((2, y_2)\) must be \(\frac{3}{4}\). Using the slope formula, we get
\[
\frac{y_2 - 6}{2 - 3} = \frac{3}{4}
\]
\[
4(y_2 - 6) = 3(-1)
\]
\[
4y_2 - 24 = -3
\]
\[
4y_2 = 21
\]
Therefore, \[y_2 = \frac{21}{4}\]