Linear Equations and Inequalities in Two Variables

Equations of Lines
Previously, we saw that the graph of any equation that can be written in the form $Ax + By = C$, where $A,B$ are not both 0 is a straight line. The equation $Ax + By = C$ is called the general form of the equation of a straight line. Moreover, we were able to write a linear equation as $y = mx + b$ by solving the equation for $y$. This form is called the slope-intercept form.

In this section, we will write an equation for a line given information about the line. Consider the following:

![Graph showing a line with points and slope]

Given:
- point (2, 3)
- slope $m = 3$

We can find an equation for the line that passes through the point (2, 3) with slope $m = 3$ by using the slope formula. Let $(x, y)$ be any point on the line except the given point (2, 3). Then

$$3 = \frac{y - 3}{x - 2} \Rightarrow 3(x - 2) = y - 3$$

Simplifying the equation, we get

$$3x - 6 = y - 3$$

$$3x - y = 3$$

which is the general equation of the line.

Generalizing the process we used to write an equation for a line gives us a formula we can use to find the equation for any line if we know its slope $m$ and a particular point on the line $(x_1, y_1)$. This form is called the point-slope form of an equation of a line.

**POINT-SLOPE FORM:**

The equation of a line with slope $m$ that passes through point $(x_1, y_1)$ is

$$y - y_1 = m(x - x_1).$$
Example 1: Write an equation for the line with slope $\frac{1}{2}$ that passes through $(2,-1)$.

Solution: Since we are given a point on the line and its slope we can use the point-slope form to write the equation.

\[ y - y_1 = m(x - x_1) \]

Substituting, we get

\[ y - (-1) = \frac{1}{2} (x - 2) \]

Multiplying by 2:

\[ 2(y + 1) = x - 2 \]

\[ x - 2y = 4 \]

We can verify what we learned earlier about a line in the form $y = mx + b$. If we know the slope $m$ and the $y$-intercept $b$, then we can use the slope-intercept form to write the equation of the line.

**SLOPE-INTERCEPT FORM:**

The equation of a line with slope $m$ and $y$-intercept $b$ is

\[ y = mx + b. \]

If we know the $y$-intercept, $b$, then the point $(0,b)$ is a point on the line. Substituting into the point-slope form, we get

\[ y - y_1 = m(x - x_1) \]

\[ y - b = m(x - 0) \Rightarrow y - b = mx \Rightarrow y = mx + b. \]

Example 2: Write an equation for the line with slope 5 and $y$-intercept 2.

Solution: Since we are given the slope 5 and $y$-intercept, 2, we can substitute in the slope-intercept form. The equation of the line is $y = 5x + 2$.

The slope-intercept form is the easiest to use, so be on the lookout for points that have an $x$-coordinate of 0.

Example 3: Write the equation of the line with slope $-1$ through the point $(0,4)$.

Solution: Although the equation can be found using the point-slope form, the point $(0,4)$ is on the $y$-axis. Therefore, the equation of the line can be written more easily by substituting into the slope-intercept form. Given $m = -1$ and $b = 4$, the equation is $y = -x + 4$.

Lines that have been written using point-slope can be simplified to slope-intercept form which is often preferred over the general form.
Horizontal Lines
For two points to line up horizontally, they must have the same $y$-value. Note that all the points on the horizontal line below will have a $y$-coordinate of 5. Therefore, the equation $y = 5$ describes the set of points graphed below.

The equation of a horizontal line through the point $(a, b)$ is $y = b$.

Example 4: What is the equation of the horizontal line through $(3, 4)$ shown below?

Answer: $y = 4$.

Vertical Lines
Similarly, points must have the same $x$-coordinate to line up vertically. The equation of a vertical line through the point $(a, b)$ is $x = a$.

Example 5: What is the equation of the vertical line shown above?

Answer: $x = 3$.

Summary:
- Given slope $m$ and point $(x_0, y_0)$, use **point-slope form**: $y - y_0 = m(x - x_0)$
- Given slope $m$ and $y$-intercept $b$: use **slope-intercept form**: $y = mx + b$
- **Horizontal line** through $(a, b)$: $y = b$
- **Vertical line** through $(a, b)$: $x = a$