Logarithmic Functions

Properties of Logarithms
We list below some of the algebraic properties of logarithms. These properties should be memorized.

1. The domain of \( \log_a x \) is all positive real numbers and its range is all real numbers.
2. \( a^{\log_a x} = x \)
3. \( \log_a(a^x) = x \)
4. \( \log_a(x^y) = y \log_a x \)
5. \( \log_a 1 = 0 \)
6. \( \log_{a^x}(a^y) = \frac{y}{x} \log_a y \)
7. \( \log_a(x \cdot y) = \log_a x + \log_a y \)
8. \( \log_a \left( \frac{x}{y} \right) = \log_a x - \log_a y \)

Proofs of the above properties:

1. This follows from the fact that the \( \log_a x \) is the inverse function of \( a^x \), and the fact that the domain of \( a^x \) is all real numbers, while its range is all positive real numbers.
2. This follows directly from the definition. That is, \( \log_a x = y \) where \( y \) is a number such that \( a^y = x \). Thus, \( x = a^y = a^{\log_a x} \).
3. This property also follows directly from the definition of \( \log_a x \). That is, \( \log_a x = y \) if and only if \( a^y = x \). Thus, \( \log_a(a^x) = y \) if and only if \( a^y = a^x \). That is, \( x = y = \log_a(a^x) \).
4. If \( x = y \), then \( \log_a x = \log_a y \), for if not, then the fact that \( a^x \) is a one-to-one function implies that \( x \neq y \). Conversely if \( \log_a x = \log_a y \), then we have \( x = a^{\log_a x} = a^{\log_a y} = y \).
5. Since \( a^0 = 1 \) no matter what \( a \) equals, we have \( \log_a 1 = 0 \) for any base \( a \).
6. Let \( z = \log_a x \), then we must have \( a^z = x \). However, we also have \( a^{\log_a x} = a^{\log_a a^{\log_a x}} = a^{\log_a x} = x \). Thus, \( a^z = a^{\log_a x} \) from which we have \( \log_a x = z = \log_a x + \log_a y \).
7. Let \( z = \log_a(x^y) \). Then \( a^z = x^y \). Now compute \( a^{\log_a x^y} = a^{\log_a (a^{\log_a x^y})} = (x)^y = x^y \). Thus, we have \( a^z = a^{\log_a x^y} \). From this we deduce that \( \log_a(x^y) = z = y \log_a x \).
8. \( \log_a \left( \frac{x}{y} \right) = \log_a(x \cdot y^{-1}) = \log_a x + \log_a (y^{-1}) = \log_a x - \log_a y \)

Example 1: Let \( f(x) = \log_2(x - 1) \). What is the domain and range of \( f(x) \)? Plot this function.
Solution: For any log function its argument must be greater than 0. Thus, domain of \( f(x) = \{x : x - 1 > 0\} = \{x : x > 1\} \). The range of \( f(x) \) = all real numbers.
Example 2: Simplify the expression \(2^{2 \log_5 5}\).

Solution:
\[
2^{2 \log_5 5} = (2^2)^{\log_5 5} = 4^{\log_5 5} = 5.
\]

Example 3: Simplify \(\log_3 16 + \log_3 5 - \log_3 8\).

Solution:
\[
\log_3 16 + \log_3 5 - \log_3 8 = \log_3 (16 \cdot 5) - \log_3 8 = \log_3 \frac{16 \cdot 5}{8} = \log_3 2 \cdot 5 = \log_3 10 \approx 2.0959.
\]

Example 4: Solve the equation \(2 + 3 \log_5 x = 15\).

Solution:
\[
2 + 3 \log_5 x = 15 \\
3 \log_5 x = 13 \\
\log_5 x = \frac{13}{3} \\
x = 5^{13/3} \approx 1068.73.
\]

Example 5: Solve the equation \(4^{x-6} = 13\).

Solution: The easiest way to solve an equation in which the unknown is part of an exponent is to take the log of both sides of the equation.
\[
4^{x-6} = 13 \quad \text{take the log}_4 \text{ of both sides} \\
x - 6 = \log_4(4^{x-6}) = \log_4 13 \quad \text{solve for } x \\
x = 6 + \log_4 13 \\
\approx 6 + 1.85022 \\
\approx 7.85.
\]
As a check on our solution we compute \(4^{7.85022-6} = 4^{1.85022} \approx 13.0\).

As we mentioned previously logarithms are the inverse functions of the exponential functions. On the next page we examine this relationship more closely for the two functions \(\log_2 x\) and \(2^x\). Since \(\log_2 x\) is the inverse function of \(2^x\) we have
\[
\log_2 x = y \text{ if and only if } x = 2^y.
\]
The following table demonstrates this relationship with specific numbers:

| $2^0$ = 1 | $\Leftrightarrow$ | $\log_2 1$ = 0 |
| $2^{-1}$ = $\frac{1}{2}$ | $\Leftrightarrow$ | $\log_2 \frac{1}{2}$ = $-1$ |
| $2^3$ = 8 | $\Leftrightarrow$ | $\log_2 8$ = 3 |
| $2^{-4}$ = $\frac{1}{16}$ | $\Leftrightarrow$ | $\log_2 \frac{1}{16}$ = $-4$ |
| $2^6$ = 64 | $\Leftrightarrow$ | $\log_2 64$ = 6 |

There is nothing special about base 2 in this relationship. For every base $a$ it is true that

$$\log_a x = y \text{ if and only if } x = a^y$$

There is another way to write these relationships, and it is

$$\log_a(a^x) = x$$

$$a^{\log_a x} = x$$

**Question:** If $a^3 = 4.56$, what is $\log_a 4.56$?

**Answer:** $\log_a 4.56 = \log_a(a^3) = 3$

The graph below contains a plot of $2^x$ and $\log_2 x$.

Notice that each plot is the reflection of the other about the line $y = x$.

**Question:** If $\log_2 15 = y$, what does $2^y$ equal?

**Answer:** 15. If $\log_2 15 = y$, then $2^y = 2^{\log_2 15} = 15$. 