Logarithmic Functions

Answers to Exercises

1. John Napier, a Scotsman.

2. In the function \( \log_2(x + 1) \) the expression \( x + 1 \) must be positive, since the domain \( \log_2 x \) is all positive real numbers. Thus, we must have

\[
\begin{align*}
  x + 1 &> 0 \\
  x &> -1.
\end{align*}
\]

3. If \( x \) is in the domain of \( \log_3(2x - 1) \), then we must have

\[
\begin{align*}
  2x - 1 &> 0 \\
  2x &> 1 \\
  x &> \frac{1}{2}
\end{align*}
\]

So the domain of \( \log_3(2x - 1) \) is \( x > \frac{1}{2} \). Moreover as \( x \) varies over the interval \( (1/2, \infty) \), \( 2x - 1 \) varies over the interval \( (0, \infty) \). This means that the range of \( \log_3(2x - 1) \) is the same as the range of \( \log_3 x \), i.e., all real numbers.

4. There are two problems in determining the domain of \( \log_5 \left(1 + \sqrt{x - 1}\right) \). One we need to ensure that

\[
1 + \sqrt{x - 1} > 0
\]

and that \( x - 1 \geq 0 \) so that we can take its square root. Since square roots are always non-negative adding 1 to \( \sqrt{x - 1} \) will certainly make \( 1 + \sqrt{x - 1} \) positive. Thus, the domain of \( \log_5 \left(1 + \sqrt{x - 1}\right) \) is

\[
x \geq 1.
\]

To determine the range of \( \log_5 \left(1 + \sqrt{x - 1}\right) \) we observe that as \( x \) varies over the interval \([1, \infty)\), the expression \( 1 + \sqrt{x - 1} \) varies over the interval \([1, \infty)\), and thus the values of \( \log_5 \left(1 + \sqrt{x - 1}\right) \) will vary over the interval \([0, \infty)\).

5. 5

6. No. \( \log_4 2 = \frac{1}{2} \) \( \log_2 4 = 2 \)

7. \( \log_{10} 15 \approx 1.17609 \)
   \( \log_3 15 \approx 1.3023 \)
   \( \log_6 15 \approx 1.51139 \)
   \( \log_4 15 \approx 1.95345 \)

As far as a relationship is concerned the smaller the base the larger the value of the logarithm.

8. \( \log_{16} 10 \approx 0.830482 \)
   \( \log_4 10 \approx 1.66096 \)
   \( \log_2 10 \approx 3.32193 \)

Notice that as we take the square root of the base the logarithm doubles.

9. \( \log_3 25 + x = 4 \)
   \( 2 + x = 4 \)
   \( x = 2 \)
10. 
\[ 5^{2x-1} = 6.3 \]
\[ 2x - 1 = \log_5 6.3 \]
\[ 2x = 1 + \log_5 6.3 \]
\[ x = \frac{1 + \log_5 6.3}{2} \]
\[ x \approx 1.0718 \]

11. \( \log_3 9 \) is larger. \( \log_3 3 = \frac{1}{2} \), \( \log_3 9 = 2 \)

12. \( 2^x = 2^{\log_2 23} = (4^{1/2})^{\log_2 23} = (4^{\log_2 23})^{1/2} \approx 4.79583 \)

13. \( x - 2 = 15^5 \)

14. \( 4^{x+2} = 15 \) implies
\[ x + 2 = \log_4 15 \]
\[ x = \log_4 15 - 2 \]
\[ x \approx 1.95345 - 2 \]
\[ x \approx -0.04655 \times 10^{-2} \]

15. \( \log_4 6 = \log_2 x \) implies
\[ x = 2^{\log_2 x} \]
\[ = 2^{\log_4 6} \]
\[ = (4^{1/2})^{\log_4 6} \]
\[ = (4^{\log_4 6})^{1/2} \]
\[ = 6^{1/2} \]

16. 1

17. To express \( 15^{x^2-3} = 9 \) as a logarithmic equation take the log to base 15 of both sides.
\[ 15^{x^2-3} = 9 \]
\[ x^2 - 3 = \log_{15} 9 \]
\[ x^2 - 3 \approx 0.811368 \]

This leads to the equation
\[ x^2 = 3.811368 \]
\[ x = \pm \sqrt{3.811368} \]
\[ x \approx \pm 1.95227 \]

18. \( \log_9 100 \) is larger, for the reason that it will take a higher power of 9 to equal 100 than for 10. In fact, these numbers are approximately
\[ \log_{10} 100 = 2.0 \]
\[ \log_9 100 \approx 2.0959 \]

19. \( \log_{10} \frac{1}{2} \)

20. 

[Graph showing logarithmic behavior with points at (25, 2) and (25, 2.3215)]
21. \( \log_4 x - \log_4 (x - 1) = 5 \) implies \( \log_4 \left( \frac{x}{x - 1} \right) = 5 \Rightarrow \frac{x}{x - 1} = 4^5 \) This leads to the equation

\[
\begin{align*}
x &= (x - 1)4^5 \\
x &= x \cdot 4^5 - 4^5 \\
x(1 - 4^5) &= -4^5 \\
x &= \frac{-4^5}{1 - 4^5} \\
&\approx 1.00098
\end{align*}
\]

22.

\[
\begin{align*}
\log_{10}(x^2) + \log_{10}(x^3) - \log_{10}(5x) &= 2 \log_{10}x + 3 \log_{10}x - \log_{10}x - \log_{10}5 \\
&= 4 \log_{10}x - \log_{10}5 \\
&= \log_{10}(x^4) - \log_{10}5 \\
&= \log_{10} \frac{x^4}{5}.
\end{align*}
\]

23.

\[
\begin{align*}
\log_2 2x - 7 \log_3 (x + 5) &= \log_2 x - \log_3 ((x + 5)^7) \\
&= \log_3 \frac{x}{(x + 5)^7}
\end{align*}
\]

24.

\[
\begin{align*}
\frac{1}{2} \log_3 16 - \log_3 24 &= \log_3 16^{1/2} - \log_3 24 \\
&= \log_3 \frac{4}{24} \\
&= \log_3 \frac{1}{6} \\
&= -\log_3 6
\end{align*}
\]

25.

\[
\begin{align*}
\log_a x - \log_a \frac{1}{x} &= \log_a x - \log_a x^{-1} \\
&= \log_a x + \log_a x \\
&= 2 \log_a x \\
&= \log_a x^2
\end{align*}
\]

26.

\[
\begin{align*}
\log_a x^3 - \log_a 3x + \log_a \frac{1}{x} &= 3 \log_a x - (\log_a 3 + \log_a x) - \log_a x \\
&= \log_a x - \log_a 3 \\
&= \log_a \frac{x}{3}
\end{align*}
\]

27.

\[
\begin{align*}
2 \log_5 x - \log_5 \sqrt{x} &= 1.3 \\
2 \log_5 x - \frac{1}{2} \log_5 x &= 1.3 \\
\frac{3}{2} \log_5 x &= 1.3 \\
\log_5 x &= \frac{2.6}{3} \\
x &= 5^{2.6/3} \\
&\approx 4.03435
\end{align*}
\]
28. 

\[ \log_3 2x + \log_3 (x^4) - 5 = 0 \]
\[ \log_3 2 + \log_3 x + 4 \log_3 x = 5 \]
\[ 5 \log_3 x = 5 - \log_3 2 \]
\[ \log_3 x = \frac{5 - \log_3 2}{5} \]
\[ = 1 - \frac{\log_2 5}{5} \]

Thus,

\[ x = 3^{1 - \frac{\log_2 5}{5}} \]
\[ = 3 \cdot 3^{-\log_2 2^{5}} \]
\[ = 3(3^{\log_2 2})^{-1/5} \]
\[ = \frac{3}{2^{1/5}} \]
\[ \approx 2.61165 \]

29. 43

30. \( \log_3 15 = \frac{\ln 15}{\ln 9} \)

31.

\[ \ln 64 \approx 4.15888 \]
\[ \ln 32 \approx 3.46574 \]
\[ \ln 16 \approx 2.77259 \]
\[ \ln 8 \approx 2.07944 \]

32.

\[ x \ln 32 = \ln 2 \]
\[ x = \frac{\ln 2}{\ln 32} \]
\[ = \frac{\ln 2}{\ln(2^5)} \]
\[ = \frac{\ln 2}{5 \ln 2} \]
\[ = \frac{1}{5} \]

33. If \( x \) is in the domain of \( \ln(x + 4) \), then we must have

\[ x + 4 > 0 \]
\[ x > -4. \]

34.

\[ \log_{a^2} x = \frac{\ln x}{\ln a^2} \]
\[ = \frac{\ln x}{2 \ln a} \]
\[ = \frac{1}{2} \log_a x \]
35. 

\[ \log_{15} x = \frac{\ln x}{\ln 15} \]
\[ = \frac{\ln 10(\log_{10} x)}{\ln 15} \]
\[ = \frac{\ln 10(3.5)}{\ln 15} \]
\[ \approx 8.05907 \]
\[ \approx 2.70805 \]
\[ \approx 2.97597 \]

36. \( \ln x \) is \( \log_e x \). If \( x > 1 \), since \( e \) is less than 5, \( \ln x \) will be larger than \( \log_5 x \). If \( 0 < x < 1 \), \( \ln x \) will be smaller than \( \log_5 x \).

37. 

\[ \log_{a^2} x = \frac{\ln x}{\ln a^2} \]
\[ = \frac{\ln x}{2 \ln a} \]
\[ = \frac{1}{2} \ln x \]
\[ = \frac{1}{2} \log_{a^2} x. \]