Polynomials

Products of Polynomials

Multiplication of Polynomials is based on the Distributive Property of Real Numbers as illustrated in the example below:

**Example 1:** Expand the following product \((x - 6)(8x + 7)\).

**Solution:**

\[
(x - 6)(8x + 7) = x(8x + 7) - 6(8x + 7) \quad \text{(treat \(8x + 7\) as a single number first)}
\]
\[
= x(8x) + x(7) - 6(8x) - 6(7)
\]
\[
= 8x^2 + 7x - 48x - 42
\]
\[
= 8x^2 - 41x - 42
\]

Notice, in the underlined step, that the distributive properties follow the familiar pattern FOIL, which is explained below.

First Outer Inner Last
terms terms terms terms

Although FOIL only works with binomials, the distributive property method works on any polynomials.

**Example 2:** Expand \((5x - 6)(x^3 - 4x^2 + 2)\).

**Solution 1**

\[
(5x - 6)(x^3 - 4x^2 + 2) = 5x(x^3 - 4x^2 + 2) - 6(x^3 - 4x^2 + 2)
\]
\[
= 5x^4 - 20x^3 + 10x - 6x^3 + 24x^2 - 12
\]
\[
= 5x^4 - 26x^3 + 24x^2 + 10x - 12
\]

(Don’t forget to distribute the negative sign as well!!!)

**Solution 2:** The distributive method is easy to organize and picture using a rectangle, or box, whose lengths are the two factors as illustrated below:

<table>
<thead>
<tr>
<th>(x^3)</th>
<th>(-4x^2)</th>
<th>+2</th>
</tr>
</thead>
<tbody>
<tr>
<td>5x</td>
<td>5(x^4)</td>
<td>-10x</td>
</tr>
<tr>
<td>-6</td>
<td>-6(x^3)</td>
<td>+24(x^2)</td>
</tr>
</tbody>
</table>

\[
= 5x^4 - 20x^3 - 6x^3 + 24x^2 + 10x - 12
\]
\[
= 5x^4 - 26x^3 + 24x^2 + 10x - 12
\]

**Example 3:** Expand \((x^2 + 9)(x^2 - 6x + 9)\).

**Solution:**

\[
(x^2 + 9)(x^2 - 6x + 9) = x^2(x^2 - 6x + 9) + 9(x^2 - 6x + 9)
\]
\[
= x^4 - 6x^3 + 9x^2 - 54x + 81
\]
\[
= x^4 - 6x^3 + 18x^2 - 54x + 81
\]
Example 4: Expand \((a + b)(a - b)\).
Solution:
\[
(a + b)(a - b) = a(a - b) + b(a - b) = a^2 - ab + ba - b^2 = a^2 - ab + ab - b^2 = a^2 - b^2
\]

The previous example \((a + b)(a - b) = a^2 - b^2\) is an example of a Special Product. Special products can not only make multiplication of certain polynomials easier, they are also useful when you want to factor a polynomial; that is, given a polynomial, find two or more polynomials whose product is the given polynomial. The most common special products are listed below (here \(a\) and \(b\) represent any numbers, variables, or algebraic expressions).

- **Difference of Squares**: \((a + b)(a - b) = a^2 - b^2\)
- **Square of a Binomial**: \((a + b)^2 = a^2 + 2ab + b^2\)
- **Cube of a Binomial**: \((a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3\)
- **Difference of Cubes**: \((a - b)(a^2 + ab + b^2) = a^3 - b^3\)
- **Sum of Cubes**: \((a + b)(a^2 - ab + b^2) = a^3 + b^3\)
- **Cube of a Binomial**: \((a - b)^3 = a^3 - 3a^2b + 3ab^2 - b^3\)

**NOTE** that there is no special product for the sum of squares that involves only real numbers.

**Historical Note**: The French mathematician Blaise Pascal (1623-1662) discovered an easy way to remember the coefficients of a binomial power using what is now called Pascal’s Triangle. After the first two rows, each successive row is obtained by adding the two numbers immediately above it as shown below:

\[
\begin{array}{c|c}
1 & \hline
1 & 1 & (a + b)^1 = 1a + 1b \\
1 & 2 & 1 & (a + b)^2 = 1a^2 + 2ab + 1b^2 \\
1 & 3 & 3 & 1 & \text{etc. } (a + b)^3 = 1a^3 + 3a^2b + 3ab^2 + 1b^3 \\
\end{array}
\]

Using this triangle, we can see that \((a + b)^4 = 1a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + 1b^4\)

The numbers in Pascal’s Triangle are also useful in Probability.

Example 5: Expand \((x - 2)^3\).
Solution:
\[
(x - 2)^3 = x^3 - 3x^2(2) + 3x(2^2) - 2^3 = x^3 - 6x^2 + 12x - 8
\]

Example 6: Expand \([(x + 2y)(x - 2y)]^2\).
Solution:
\[
[(x + 2y)(x - 2y)]^2 = [x^2 - (2y)^2]^2 \text{ (work inside the parenthesis first)}
= [x^2 - 4y^2]^2 \\
= (x^2)^2 - 2(x^2)(4y^2) + (4y^2)^2 \\
= x^4 - 8x^2y^2 + 16y^4
\]