Polynomial Functions

Fundamental Theorem of Algebra

In this page we state the Fundamental Theorem of Algebra. This result was first proven by Carl Friedrich Gauss, who was born in Germany in 1777 and died in 1855. In fact, during his life he gave three different proofs of the Fundamental Theorem of Algebra.

Before stating this theorem we need to define what we mean by the multiplicity of a root, and we do this first with an example.

Example 1: Let \( p(x) = (x - 1)^3(x - 3)^4(x + 5) \). It is clear that \( p(x) \) has three distinct roots 1, 3, and \(-5\). However, roots 1 and 3 appear more than once. The root 1 appears 3 times and the root 3 appears 4 times, while the root \(-5\) appears just once. We say that 1 is a root of multiplicity 3, 3 is a root of multiplicity 4, and \(-5\) is a root of multiplicity 1.

Definition: Let \( p(x) \) be a polynomial. We say that \( c \) is a root of multiplicity \( k \), where \( k \) is a positive integer, if \( c \) is a root of \( p(x) \), and if \( (x - c)^k \) divides \( p(x) \) while \( (x - c)^{k+1} \) does not divide \( p(x) \).

We are now ready to state the Fundamental Theorem of Algebra:

Theorem: Let \( p(x) \) be a polynomial of degree \( n \) with real or complex coefficients. Then, counting multiplicities, \( p(x) \) has exactly \( n \) roots.

Example 2: Let \( p(x) = (x - 1)^3(x - 3)^4(x + 5) \). Then
\[
p(x) = x^8 - 10x^7 + 18x^6 + 158x^5 - 956x^4 + 2274x^3 - 2754x^2 + 1674x - 405.
\]
A polynomial of degree 8. According to the Fundamental Theorem of Algebra, \( p(x) \) must have 8 roots. (Remember we count the multiplicities of the roots.) From the factored form of \( p(x) \) we see

<table>
<thead>
<tr>
<th>Root</th>
<th>Multiplicity</th>
</tr>
</thead>
<tbody>
<tr>
<td>(-5)</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>sum of multiplicities</td>
<td>8</td>
</tr>
</tbody>
</table>

The sum of the multiplicities of this polynomial equals 8, the degree of the polynomial.

Note: if complex numbers are not introduced then the Fundamental Theorem of Algebra is not correct. An example of this is the polynomial \( x^2 + 1 \). This polynomial has no real roots. It does have two roots, but both are complex.

Question: How many roots does the polynomial \( p(x) = -2x^2 + 5x^6 - 8x + 7 \) have?
Answer: This polynomial has 6 roots.