Polynomial Functions

Answers to Exercises

1.  13
2.  3
3.  We know that \( p(x) = -3x^2 + bx + 8 \). We also know that \( 5 = p(1) = -3 + b + 8 = b + 5 \). This implies that \( b = 0 \), and hence, \( p(x) = -3x^2 + 8 \).
4.  Items a, c, and e are polynomials.
5.  The leading coefficient is 7 and the constant term is 4.
6.  a. 4  b. 1  c. 45  d. 5
7.  For large values of positive \( x \) the values of the polynomial \( x^3 \) get very large positively. For large negative values of \( x \) the polynomial \( x^3 \) also gets large in a negative sense.
8.  The values of \( p(x) \) go to \( -\infty \) as \( x \) goes to \( +\infty \).
9.  No, since \( p(x) \) behaves the same as \( x \) get large in either direction, the degree of \( p(x) \) must be even. But, if the leading coefficient of this polynomial is negative, then the polynomial’s values must go to minus infinity, not plus infinity.
10. \( \frac{3x^4 - 5x + 7}{x^2} = 3x^2 + \frac{7 - 5x}{x^2} \). The remainder is \( -5x + 7 \).
11. \( \frac{6x^4 - 3x^3 + x - 5}{2x^2 + x} = 3x^2 - 3x + \frac{1}{2} + \frac{-5 - \frac{1}{2}x}{2x^2 + x} \). Hence, the quotient is \( 3x^2 - 3x + \frac{1}{2} \) and the remainder is \( -\frac{x}{2} - 5 \).
12. \( \frac{x^3 - 8}{x - 2} = x^2 + 2x + 4 \)
13. \( \frac{x^3 + 1}{x + 1} = x^2 - x + 1 \)
14. The leading coefficient and degree of the quotient are \( \frac{5}{3} \) and 3 respectively.
15. \( \frac{x^4 - 2x + 3}{x - 1} = x^3 + x^2 + x - 1 + \frac{2}{x-1} \)
16. \( \frac{3x^7 + x^6 - 3x^2 + x - 5}{2x^3 - 7} = \frac{3}{2}x^4 + \frac{1}{2}x^3 + \frac{21}{4}x + \frac{7}{4} + \frac{29}{4} - 3x^2 + \frac{151}{4}x \)
17. \( \frac{4x^7 - 6x^4 - 19}{2x^3 - 5x + 3} = \frac{2x^4 + 5x^2 - 6x + 25}{2} + \frac{-113}{2} - 45x^2 + \frac{161}{2}x \). Thus, the quotient equals \( 2x^4 + 5x^2 - 6x + \frac{25}{2} \).
18. No.
19. A sign table for \((x - 1)^2(x - 2)^3\) is

<table>
<thead>
<tr>
<th>( x )</th>
<th>((x - 1)^2)</th>
<th>((x - 2)^3)</th>
<th>((x - 1)^2(x - 2)^3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x &lt; 1 )</td>
<td>( &gt; 0 )</td>
<td>( &lt; 0 )</td>
<td>( &lt; 0 )</td>
</tr>
<tr>
<td>( 1 &lt; x &lt; 2 )</td>
<td>( &gt; 0 )</td>
<td>( &lt; 0 )</td>
<td>( &lt; 0 )</td>
</tr>
<tr>
<td>( 2 &lt; x )</td>
<td>( &gt; 0 )</td>
<td>( &gt; 0 )</td>
<td>( &gt; 0 )</td>
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</tbody>
</table>

20. \( p(x) = -13(x - 6)^3(x + 11)^2 \)
21. The possible rational roots of $3x^3 - x^2 - 6x + 2$ are

$$\left\{ \pm 1, \pm 2, \pm \frac{1}{3}, \pm \frac{2}{3} \right\}$$

An easy check shows that $\frac{1}{3}$ is a root. Thus, $(x - 1/3)$ is a factor of $p(x)$.

$$p(x) = 3x^3 - x^2 - 6x + 2$$
$$= (x - \frac{1}{3})(3x^2 - 6)$$
$$= 3(x - 1/3)(x^2 - 2)$$

The other two roots of $p(x)$ are $\pm \sqrt{2}$. Since we found one root of a third degree polynomial, that meant we could factor $p(x)$ as a product of a linear term (a polynomial of degree one) and a quadratic term (a polynomial of degree two). We can then either find the roots of the quadratic term by inspection, or we can use the quadratic formula.

22. Remember that $c$ is a root of a polynomial $p(x)$ if and only if $x - c$ divides $p(x)$. Check that 2 is a root:

$$\begin{array}{rrrrr} 
1 & 0 & -13 & 0 & 36 \\
2 & 4 & -18 & -36 \\
1 & 2 & -9 & -18 & 0 \\
\end{array}$$

Check that $-2$ is a root:

$$\begin{array}{rrrrr} 
1 & 0 & -13 & 0 & 36 \\
-2 & 4 & 18 & -36 \\
1 & -2 & -9 & 18 & 0 \\
\end{array}$$

23. Remember, according to the remainder theorem $p(c)$ equals the remainder obtained when $p(x)$ is divided by $x - c$.

$$c = 1: \begin{array}{rrrrr} 
1 & -5 & 0 & 0 & 2 & -7 \\
-5 & -5 & -5 & -3 & -10 \\
\end{array} \quad p(1) = -10.$$  

$$c = 3: \begin{array}{rrrrr} 
1 & -5 & 0 & 0 & 2 & -7 \\
-15 & -45 & -135 & -399 \\
\end{array} \quad p(3) = -406.$$  

$$c = -2: \begin{array}{rrrrr} 
-2 & -5 & 0 & 0 & 2 & -7 \\
10 & -20 & 40 & -84 \\
-5 & 10 & -20 & 42 & -91 \\
\end{array} \quad p(-2) = -91.$$  

$$c = -3: \begin{array}{rrrrr} 
-3 & 1 & -4 & -10 & 40 & 9 & -36 \\
-3 & 21 & -33 & -21 & 36 \\
\end{array} \quad p(-3) = 0.$$  

$$c = 1: \begin{array}{rrrrr} 
1 & -4 & -10 & 40 & 9 & -36 \\
1 & -3 & -13 & 27 & 36 & 0 \\
\end{array} \quad p(1) = 0.$$  

$$c = 4: \begin{array}{rrrrr} 
4 & 1 & -4 & -10 & 40 & 9 & -36 \\
4 & 0 & -40 & 0 & 36 \\
\end{array} \quad p(4) = 0.$$  

24. If $\frac{r}{s}$ is a rational root of $6x^4 - 5x^3 + 1$, then $r$ must divide 1 and $s$ must divide 6. Thus,

$$\frac{r}{s} = \pm \left\{ 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{6} \right\}$$
26. The possible rational roots of \( x^5 - 15x^3 + 10x^2 + 60x - 72 \) are the integer divisors of 72. After checking the possible roots the values 2 and \(-3\) are found to be roots.
\[ p(x) = (x - 2)(x + 3)(x^3 - x^2 - 8x + 12). \]
The possible rational roots of the last factor are
\[ \pm \left\{1, 2, 3, 4, 6, 12\right\}. \]

Checking the possibilities we verify that \( x = 2 \) and \( x = -3 \) are roots of this cubic. So
\[ p(x) = (x - 2)(x + 3)[(x - 2)(x + 3)(x - 2)] \]
\[ = (x - 2)^3(x + 3)^2 \]
Thus, 2 is a root of multiplicity 3 and \(-3\) is a root of multiplicity 2.

27. If \( p(x) \) is the polynomial then
\[ p(x) = (x + 5)^7q(x) \]
where \( q(x) \) is any polynomial.

28. Since the only powers of \( x \) which appear in \( p(x) = 6x^4 - 5x^2 + 1 \) are even if \( c \) is a root so is \(-c\). Thus, we only need check the possible rational roots which are also positive. \( p(1) = 2, p(1/2) = \frac{1}{8}, p(1/3) = \frac{14}{27}, p(1/6) = \frac{187}{216} \). Thus, this polynomial has no rational roots.

29. Notice that \( p(x) = 6x^4 - 5x^2 + 1 \) is a quadratic in \( x^2 \). Thus, we can use the quadratic formula to determine what \( x^2 \) must equal.
\[ x^2 = \frac{5 \pm \sqrt{25 - 24}}{12} \]
\[ = \frac{5 \pm 1}{12} \]
\[ = \frac{1}{2} \text{ or } \frac{1}{3}. \]

Thus,
\[ x = \pm \frac{1}{\sqrt{2}} \text{ or } \pm \frac{1}{\sqrt{3}}. \]

30. The possible rational roots of \( p(x) = 12x^4 - 23x^3 + x^2 + 9x - 2 \) are
\[ \pm \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{6}, \frac{1}{12}, 2, \frac{2}{3}\right\}. \]

After checking them, use synthetic division, we find that \( \frac{1}{4} \) and \( \frac{2}{3} \) are rational roots. Thus, we can factor \( p(x) \)
\[ p(x) = (x - 1/4)(x - 2/3)(12x^2 - 12x - 12) \]
\[ = 12(x - 1/4)(x - 2/3)(x^2 - x - 1) \]
The roots of the quadratic term are
\[ \frac{1}{2} \sqrt{5} + \frac{1}{2} \text{ and } \frac{1}{2} - \frac{1}{2} \sqrt{5} \]
The quadratic formula was used to find these two roots.

31. If \( \frac{c}{a} \) is a rational root, then
\[ \frac{c}{a} = \pm \left\{1, 3, \frac{1}{2}, \frac{1}{3}\right\}. \]

We use a calculator to see if any of these are roots.
\[ p(1) = -4, \quad p(-1) = 0, \quad p(3) = 120, \quad p(-3) = 180 \]
\[ p(1/2) = -\frac{15}{4}, \quad p(-1/2) = -\frac{5}{2}, \quad p(3/2) = 0, \quad p(-3/2) = \frac{39}{4}. \]

So our polynomial has two rational roots. They are \(-1\) and \(3/2\).

32. The roots are \(-1\) and \(3/2\). Thus, \((x + 1)\) and \((x - 3/2)\) must divide \( p(x) = 2x^4 - x^3 - x^2 - x - 3. \)
\[ \frac{2x^4 - x^3 - x^2 - x - 3}{(x + 1)(x - 3/2)} = 2x^2 + 2. \]
Thus,
\[ p(x) = 2(x^2 + 1)(x + 1)(x - 3/2). \]
The roots are \( \pm 1, 2, 5, 10 \).

Thus, 5 is a possible root, and -3 is not.

34. The only possible rational roots are \( \pm 1 \). Since neither \( p(1) \) nor \( p(-1) \) is zero, this polynomial has no rational roots.

35. There are none.

36. 2

37. \( p(-x) = -x^5 + x^4 + x + 2 \) has 1 sign change.

38. \( p(x) = x^5 + x^4 - x + 2 \) has 2 sign changes. Thus, \( p(x) \) has either 2 or 0 positive real roots.

39. The polynomial \( x^3 - x - 1 \) has 1 sign change. Thus, it must have 1 positive real root.

40. \( p(x) = x^6 - x^4 - x^3 - x - 2 \) has one sign change, hence, there is exactly one positive real root.

41. If \( p(x) = x^4 - 5x^3 - 8 \), then \( p(-1) = -12 \) and \( p(3) = 28 \). Thus, \( p(x) \) must have a root somewhere between -1 and 3.

\[
\begin{array}{|c|c|}
\hline
x & p(x) \\
\hline
-1 & 2 \\
-1/2 & 0.96875 \\
-1/4 & -9.7656 \times 10^{-4} \\
\hline
\end{array}
\]

From this table we see that there is a root in the interval \([-1/2, -1/4]\). In fact the root is pretty close to \( x = -1/4 \), since \( p(-1/4) \) is of order of \( 10^{-4} \).

\[
\begin{array}{|c|c|}
\hline
x & p(x) \\
\hline
1 & -4.0 \\
3/2 & 0.59375 \\
5/4 & -2.9482 \\
11/8 & -1.5851 \\
\hline
\end{array}
\]

From the last row of the table we see that \( p(x) \) has a root in the interval \([11/8, 3/2]\).

42. Since \( p(3) = -8 \) and \( p(4) = 8 \), the intermediate value theorem implies that there must be a root somewhere between 3 and 4.

43. Yes, \( p(1) = -1 \), and \( p(2) = 11 \).

44. Absolutely nothing can be inferred about the roots. Since \( p(0) = 2 \), we know that the constant term of the polynomial equals 2.

45. The roots are \( x = 4 \), and \( x = \pm \sqrt{5} i \)

46. This is hard unless we can find a root or two to factor the polynomial. Let’s try to find a rational root. The possibilities are

\[ \pm 1, 2, 5, 10 \].

It turns out that 1 is a root. Thus,

\[
x^4 + 3x^3 + 5x^2 + x - 10 = (x - 1)(x^3 + 4x^2 + 9x + 10)
\]

Descarte’s rule of signs tells us that the second factor has no positive real roots. So let’s see if we can find a negative rational root. We get lucky again –2 is a root. Synthetic division was used to check all of the negative rational roots. In any case we now know that \( x + 2 \) is a factor, and we have

\[
x^4 + 3x^3 + 5x^2 + x - 10 = (x - 1)(x^3 + 4x^2 + 9x + 10) = (x - 1)(x + 2)(x^2 + 2x + 5)
\]

We now use the quadratic formula to find the remaining two roots of this polynomial.
\[ x = \frac{-2 \pm \sqrt{2 - 20}}{2} = \frac{-2 \pm 3\sqrt{2}i}{2}. \]

The roots are
\[ 1, \ -2, \ -1 + \frac{3}{\sqrt{2}}i, \ -1 - \frac{3}{\sqrt{2}}i. \]

Notice that the complex roots are in conjugate pairs.

49. The roots of \( x^2 + ix \) are 0 and \(-i\). The fact that \( i \), the conjugate of \(-i\), is not a root does not contradict the statement that complex roots come in pairs because the coefficients of the polynomial \( x^2 + ix \) are not all real numbers.

50. Since \( \sqrt{3}i \) is a root of \( x^4 + 7x^3 + 2x^2 + 6x + 12 \), then so is \(-\sqrt{3}i\). Thus, \((x - \sqrt{3}i)(x + \sqrt{3}i) = (x^2 + 3)\) is a factor of the polynomial, and \(\frac{x^4 + 7x^3 + 2x^2 + 6x + 12}{x^2 + 3} = x^2 + 2x + 4\). The roots of \( x^2 + 2x + 4 \) are
\[ \frac{-2 \pm \sqrt{4 - 16}}{2} = -1 \pm \sqrt{3}i. \]

The roots of this polynomial are
\[ -1 \pm \sqrt{3}i \] and \( \pm \sqrt{3}i \)

51. The polynomial \( p(x) = x^3 - 3x + 1 \) has two sign changes. Thus, it must have 2 or 0 positive real roots. Since \( p(0) = 1 \) and \( p(1) = -1 \) we know there is a root between 0 and 1. Thus, we must have a second positive root.

52. If \( p(x) = x^3 - 3x + 1 \), then \( p(-x) = -x^3 + 3x + 1 \). This last expression has 1 sign change. Thus, \( p(x) \) must have exactly 1 negative root.

53. 4

54. 2

55. \( p(x) = x^4 + 7x^3 + 2x^2 + 6x + 12 \) and \( p(-x) = x^4 + 7x^3 - 2x^2 - 6x + 12 \). Thus, using Descarte’s rule of signs, this polynomial has no positive real roots and either 2 or 0 negative real roots. Since 0 is not a root the polynomial must have either 2 or 4 complex roots. As we saw in a previous exercise there are 4 complex roots.