Quadratic Functions

Definition

In this and the next few pages we will define what a quadratic function is, and examine some of their properties. We start by defining what a quadratic function is.

**Definition:** A quadratic function is any function of the form \( f(x) = ax^2 + bx + c \), where the constant \( a \neq 0 \). If \( a = 0 \) we have a linear function, not a quadratic one.

A few particular examples and their graphs are shown below.

**Example 1:** Let \( f(x) = x^2 \). Compute \( f(-2), f(2), \) and \( f(3) \).

Solution: \( f(-2) = (-2)^2 = 4 \), \( f(2) = 2^2 = 2 \), and \( f(3) = 3^2 = 9 \). A plot of \( f(x) = x^2 \) is shown below. Notice that the curve opens up, and the coefficient of \( x^2 \) is 1, a positive number.

![Graph of \( f(x) = x^2 \)](image)

**Example 2:** Let \( f(x) = -x^2 + 5 \). Compute \( f(-2), f(2), \) and \( f(3) \).

Solution: \( f(-2) = -(2)^2 + 5 = 1 \), \( f(2) = -(2)^2 + 5 = 1 \), and \( f(3) = -(3)^2 + 5 = -4 \). A plot is shown below. Notice that this curve opens down, and that the coefficient of \( x^2 \) is -1, a negative number.

![Graph of \( y = -x^2 + 5 \)](image)

The above two graphs of quadratic functions are for all practical purposes the only different graphs possible. That is, the curve of any quadratic function either opens up \( (a > 0) \) or opens down \( (a < 0) \). Where the vertex is located, and how sharply the curve opens up around the vertex will of course vary from one quadratic function to the next, and depends upon the values of both \( a, b, \) and \( c \).
Example 3: Let \( f(x) = x^2 - 3x - 4 \). Plot this quadratic function, determine whether it opens up or down, and the location of its vertex.

Solution: Since the coefficient of \( x^2 \) is 1 and is positive, we expect the curve to open up. Its plot is shown below.

\[
y = x^2 - 3x - 4
\]

It’s hard to tell from the graph exactly where the vertex is located. It does appear to be located between 1 and 2. Maybe 1.5 or 1.6.

We can determine the precise location of the vertex by rewriting the function. The process used, which we have seen earlier, is completing the square.

\[
x^2 - 3x - 4 = x^2 - 3x + \frac{9}{4} - \frac{9}{4} - 4
\]

\[
= \left(x - \frac{3}{2}\right)^2 - \frac{9}{4} - 4
\]

\[
= \left(x - \frac{3}{2}\right)^2 - \frac{25}{4}
\]

**Question:** Why did we choose \( \frac{9}{4} \)?

**Answer:** We chose to add and subtract \( \frac{9}{4} \), because one-half of the coefficient of \( x \) is \( -\frac{3}{2} \), and we need to add and subtract the square of \( -\frac{3}{2} \). Remember \( (x + b)^2 = x^2 + 2bx + b^2 \), and the constant term \( b^2 \) is the square of one-half of the coefficient of \( x \).

So how does rewriting the function, as we have done, enable us to determine the location of the vertex. Well, the vertex is located at that point where the function takes on its smallest value. (If the graph opens downward, then the vertex is located at the point where the function takes on its largest value.) We have written the function as the sum of two terms. The first term \( \left(x - \frac{3}{2}\right)^2 \) depends on \( x \) and is never negative. Thus, the smallest value the first summand can have is zero, which occurs when \( x = \frac{3}{2} \). The second term \( -\frac{25}{4} \) does not change when we vary \( x \). Thus, the smallest value of the function occurs when the first term is zero. That is, when \( x = \frac{3}{2} \), and in that case \( f\left(\frac{3}{2}\right) = -\frac{25}{4} \).

A more mathematical appearing argument follows:

\[
x^2 - 3x - 4 = \left(x - \frac{3}{2}\right)^2 - \frac{25}{4}
\]

\[
\geq 0 - \frac{25}{4}
\]

\[
= -\frac{25}{4}.
\]
Example 4: Graph the quadratic function \( f(x) = -3x^2 + x + 4 \). Show that it opens downward, and locate its vertex.

Solution: Since the coefficient of \( x^2 \) is \(-3\), the graph has to open downward. To locate its vertex we complete the square as we did in the previous example.

\[
-3x^2 + x + 4 = -3\left(x^2 - \frac{1}{3}x\right) + 4
\]
\[
= -3\left(x^2 - \frac{1}{3}x + \frac{1}{36}\right) + \frac{1}{12} + 4
\]
\[
= -3\left(x - \frac{1}{6}\right)^2 + \frac{49}{12}
\]

Question: Why did we add the term \( \frac{1}{12} \)?

Answer: We introduced the term \( \frac{1}{36} \) into the factor \( \left(x^2 - \frac{1}{3}x\right) \) in order to make it a perfect square. However, that term is multiplied by \(-3\). Thus, we actually subtracted \( 3 \cdot \frac{1}{36} = \frac{1}{12} \). So, to balance the \( \frac{1}{12} \) which we subtracted and preserve the equality, we have to add \( \frac{1}{12} \).

From this representation for \( f(x) \), i.e., \( f(x) = -3\left(x - \frac{1}{6}\right)^2 + \frac{49}{12} \), we see that the largest value of this function occurs when \( x = \frac{1}{6} \) and this maximum value equals \( \frac{49}{12} \).