Exponents and Radicals

Radicals and Properties of Radicals
Radicals (or roots) are, in effect, the opposite of exponents. In other words, the \( n \)th root of a number \( a \) is a number \( b \) such that
\[
b = \sqrt[n]{a} = a^{\frac{1}{n}} \iff b^n = a
\]
The number \( b \) is called an \( n \)th root of \( a \). The number \( n \) is referred to as the index of the radical (if no index appears, \( n \) is understood to be 2). The principal \( n \)th root of a number is the \( n \)th root of \( a \) which has the same sign as \( a \). For example both 2 and \(-2\) satisfy \( x^2 = 4 \), but 2 is the (principal) square root of 4.

Examples:
- \( \sqrt{27} = 3 \) since \( 3^3 = 27 \)
- \( \sqrt{16} = 2 \) since \( 2^4 = 16 \) (Note \(-2)^4 = 16 \) also, but 2 is the principal \( 4 \)th root)
- \( \sqrt[3]{-64} = -4 \) since \(-4)^3 = -64 \)
- \( \sqrt[3]{-81} \) is not a real number and we will say that it does not exist. (In this course we won’t learn how to take an even \( n \)th power of a negative number.)

Radicals are used to define rational exponents:
\[
a^{\frac{1}{n}} = \sqrt[n]{a}
\]
\[
a^{\frac{m}{n}} = (\sqrt[n]{a})^m
\]
The notation \( a^{\frac{1}{n}} \) is extremely useful, and we encourage you to use it whenever you have to simplify expressions involving radicals.

Examples:
- \( (125)^{\frac{1}{3}} = \sqrt[3]{125} = 5 \)
- \( (-64)^{\frac{1}{3}} = \sqrt[3]{(-64)^2} = (\sqrt[3]{-64})^2 = (-4)^2 = 16 \)
- \( (32)^{\frac{1}{5}} = \frac{1}{(32)^{\frac{1}{5}}} = \frac{1}{(\sqrt[5]{32})^3} = \frac{1}{2^3} = \frac{1}{8} \)

Since radicals are nothing more than rational exponents, many of the properties of exponents also apply to radicals.

<table>
<thead>
<tr>
<th>Property</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 ( \sqrt{a^m} = (\sqrt{a})^m )</td>
<td>( \sqrt[3]{32^3} = (\sqrt[3]{32})^3 = 2^3 = 8 )</td>
</tr>
<tr>
<td>2 ( \sqrt{a} \cdot \sqrt{b} = \sqrt{ab} )</td>
<td>( \sqrt{27} \cdot \sqrt{3} = \sqrt{81} = 9 )</td>
</tr>
<tr>
<td>3 ( \frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}} , b \neq 0 )</td>
<td>( \sqrt{\frac{27}{8}} = \frac{\sqrt{27}}{\sqrt{8}} = \frac{3}{2} )</td>
</tr>
<tr>
<td>4 ( \sqrt[4]{a} = a^{\frac{1}{4}} )</td>
<td>( \sqrt[4]{2x} = \sqrt[4]{2x} )</td>
</tr>
<tr>
<td>5a If ( n ) is odd ( \sqrt[n]{a^m} = a )</td>
<td>( \sqrt[3]{(-127)^3} = -127 )</td>
</tr>
<tr>
<td>5b If ( n ) is even ( \sqrt[n]{a^m} =</td>
<td>a</td>
</tr>
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</table>
The following list is a restatement of these properties, but in exponential notation. You need to be familiar with both radical and exponential notation, and be able to convert between the two.

<table>
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<tr>
<td>1 ((a^m)^{1/n} = a^{m/n})</td>
<td>((32^3)^{1/5} = (32^{1/5})^3 = 2^3 = 8)</td>
</tr>
<tr>
<td>2 (a^{1/n}b^{1/n} = (ab)^{1/n})</td>
<td>(27^{1/2}3^{1/2} = (27 \cdot 3)^{1/2} = 81^{1/2} = 9)</td>
</tr>
<tr>
<td>3 (a^{1/n}b^{1/n} = \left(\frac{a}{b}\right)^{1/n}, b \neq 0)</td>
<td>(\left(\frac{27}{8}\right)^{1/3} = \frac{27^{1/3}}{8^{1/3}} = \frac{3}{2})</td>
</tr>
<tr>
<td>4 ((a^{1/m})^{1/n} = a^{1/mn})</td>
<td>((2x)^{1/4} = (2x)^{1/8})</td>
</tr>
<tr>
<td>5a If (n) is odd ((a^n)^{1/n} = a)</td>
<td>((-127)^3)^{1/3} = -127)</td>
</tr>
<tr>
<td>5b If (n) is even ((a^n)^{1/n} =</td>
<td>a</td>
</tr>
</tbody>
</table>

Examples:
- \(\sqrt[3]{-3}^2 = 3\) (refer to Property 5b)
- \(16^{3/2} = 16^{3/2} = ((16)^{1/2})^3 = 4^3 = 64\) (refer to property 1 given the right hand side)
- \((-16)^{3/2} = (-16)^{3/2} = ((-16)^{1/2})^3 = (-16)^{1/2} = \sqrt{-16}^3\) (refer to property 1)
  
  There is no answer as we cannot take the square root of \(-16\).
- \(32^{1/5}(27)^{1/3} = (32^{1/5})(27)^{1/3} = 2 \cdot 3 = 6\)