Rationalizing Denominators

Rationalizing the Denominator is a technique used whenever a radical appears in the denominator of an expression. To begin with, make sure the radical is simplified; that is, perfect roots are pulled out. The next step depends on how the radical appears in the denominator:

I. The Radical is a single term

Multiply both the numerator and the denominator of the expression by something which will produce a perfect root in the denominator.

Example 1: Rationalize the denominator: \( \frac{8}{\sqrt[3]{54}} \)

Solution: First simplify the radical in the denominator:

\[
\frac{8}{\sqrt[3]{54}} = \frac{8}{\sqrt[3]{2 \cdot 3^3}} = \frac{8}{3\sqrt[3]{2}} \quad \text{(Recall that } \sqrt[3]{3^3} = 3) \]

Note that if the 2 was a cube, we would no longer need a radical in the denominator.

To rationalize this denominator, we multiply both the numerator and denominator by \( \sqrt[3]{2^2} \) as follows:

\[
\frac{8 \cdot \sqrt[3]{2^2}}{3\sqrt[3]{2} \cdot \sqrt[3]{2^2}} = \frac{8\sqrt[3]{4}}{3 \cdot 2} = \frac{4\sqrt[3]{4}}{3}
\]

**Question:** Why do we pick \( \sqrt[3]{2^2} \)?

**Answer:** We want to multiply the denominator of \( \frac{8}{3\sqrt[3]{2}} \) by a term which will enable us to remove any radicals. Thus, we need to figure out what to multiply \( \sqrt[3]{2} \) by. Well, if we write this in exponential notation, we have

\[
\sqrt[3]{2} = 2^{1/3}.
\]

Clearly if we multiply by \( 2^{2/3} \) we get something nice.

\[
2^{1/3} \cdot 2^{2/3} = 2^1 = 2.
\]

Moreover \( 2^{2/3} = \sqrt[3]{2^2} \).

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**Example 2:** Rationalize the denominator: \( \frac{5}{\sqrt{10}} \)

Solution: \( \frac{5}{\sqrt{10}} \cdot \frac{\sqrt{10}}{\sqrt{10}} = \frac{5\sqrt{10}}{10} = \frac{\sqrt{10}}{2} \)

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**Example 3:** Rationalize the denominator: \( \frac{1}{\sqrt{(5x)^2}} \)

Solution: \( \frac{1}{\sqrt{(5x)^2}} \cdot \frac{\sqrt{5x}}{\sqrt{5x}} = \frac{\sqrt{5x}}{\sqrt{(5x)^2}} = \frac{\sqrt{5x}}{5x} \) (note that you only need one more 5x)

\[
= \frac{\sqrt{5x}}{5x}
\]
II. The Denominator is a Sum of Terms

In this example the denominator has the form $a + b\sqrt{m}$ (here we only concern ourselves with square roots). The conjugate of this expression is $a - b\sqrt{m}$. In the example on the next page, notice what happens when we multiply the numerator and denominator by the conjugate:

\[
\frac{5}{5 + \sqrt{7}} \cdot \frac{5 - \sqrt{7}}{5 - \sqrt{7}} = \frac{5(5 - \sqrt{7})}{(5 + \sqrt{7})(5 - \sqrt{7})}
\]

\[
= \frac{25 - 5\sqrt{7}}{5^2 - (\sqrt{7})^2}
\]

\[
= \frac{25 - 5\sqrt{7}}{25 - 18}
\]

Here the multiplication in the denominator is done using a distributive property technique called FOIL (shown below). You can also use the special product $(a + b)(a - b) = a^2 - b^2$ which we will discuss in detail in another section.

\[
(5 + \sqrt{7})(5 - \sqrt{7})
= (5 \cdot 5) + 5(-\sqrt{7}) + 5\sqrt{7} + (\sqrt{7})(-\sqrt{7})
\]

First Outer Inner Last

\[
= 5^2 - 5\sqrt{7} + 5\sqrt{7} - (\sqrt{7})^2
\]

\[
= 5^2 - (\sqrt{7})^2
\]

Note how the inner terms cancel.

\[
= 25 - 7 = 18
\]

**Question:** What should you multiply $5 - \sqrt{3}$ by to rationalize it?

**Answer:** Multiply $5 - \sqrt{3}$ by $5 + \sqrt{3}$.

\[
(5 - \sqrt{3})(5 + \sqrt{3}) = 25 - 9 = 16.
\]

Remember

\[
(a - b)(a + b) = a^2 - b^2.
\]

**Question:** What should you multiply $a + b\sqrt{c}$ by to rationalize it?

**Answer:** Multiply $a + b\sqrt{c}$ by $a - b\sqrt{c}$.

\[
(a + b\sqrt{c})(a - b\sqrt{c}) = a^2 - b^2(\sqrt{c})^2 = a^2 - b^2|c|.
\]

**Example 4:** Rationalize the denominator: $\frac{2x}{5 - \sqrt{3}}$

**Solution:**

\[
\frac{2x}{5 - \sqrt{3}} \cdot \frac{5 + \sqrt{3}}{5 + \sqrt{3}} = \frac{2x(5 + \sqrt{3})}{5^2 - (\sqrt{3})^2}
\]

\[
= \frac{2x(5 + \sqrt{3})}{25 - 3}
\]

\[
= \frac{2x(5 + \sqrt{3})}{22}
\]

\[
= \frac{x(5 + \sqrt{3})}{11}
\]
Example 5: Rationalize the denominator: \( \frac{4}{\sqrt{5} + \sqrt{6}} \)

Solution: \[
\frac{4}{\sqrt{5} + \sqrt{6}} \cdot \frac{\sqrt{5} - \sqrt{6}}{\sqrt{5} - \sqrt{6}} = \frac{4(\sqrt{5} - \sqrt{6})}{(\sqrt{5})^2 - (\sqrt{6})^2}
\]
\[
= \frac{4\sqrt{5} - 4\sqrt{6}}{-11} \text{ or } -\frac{4\sqrt{5} - 4\sqrt{6}}{11}
\]