Rational Functions

Definition of a Rational Function

In this page we define a rational function and look at some graphs of rational functions.

**Definition:** A rational function is the quotient of two polynomials. That is, if $r(x)$ is a rational function, then there are two polynomials $p(x)$ and $q(x)$ such that $r(x) = \frac{p(x)}{q(x)}$.

**Example 1:** If $p(x) = 3x - 5$ and $q(x) = x^2 - 2x$, then $r(x) = \frac{3x - 5}{x^2 - 2x}$. The quotient of $p(x)$ and $q(x)$, is a rational function of $x$. A plot of $\frac{3x - 5}{x^2 - 2x}$ is shown below. Notice the behavior of $r(x)$ for $x$ close to 0 or 2. We say that the function has vertical asymptotes at $x = 0$ and at $x = 2$.

The domain of a rational function is the set of real numbers $x$ for which one can compute $\frac{p(x)}{q(x)}$. Since polynomials are defined for all $x$, the only real numbers which are not in the domain of any rational function are those values of $x$ for which the denominator is zero. Thus, the domain of any rational function $\frac{p(x)}{q(x)}$ is the set of $x$ for which $q(x) \neq 0$.

If we look at the rational function in the above example, its denominator equals $x^2 - 2x = x(x - 2)$. Thus, the numbers $x = 0$ and $x = 2$ (The values for which the denominator equals zero.) are not in the domain of the function $\frac{3x - 5}{x^2 - 2x}$, and all other values of $x$ are in the domain.

The $x$-intercepts of a rational function $r(x)$ are those real numbers $x$ for which $r(x) = 0$. Note that a fraction can equal zero only if its numerator is zero. Thus, the $x$-intercepts are those values of $x$ for which the numerator $p(x) = 0$.

The $y$-intercept is that number $y_0$ such that $r(0) = y_0$.

Notice that there may be many $x$-intercepts, but there is at most one $y$-intercept.

**Question:** When will there be no $y$-intercept?

**Answer:** When 0 is not in the domain of the rational function.
In the previous example \( r(x) = \frac{3x - 5}{x^2 - 2x}, \) \( x = 0 \) is not in the domain of this rational function, hence there is no \( y \)-intercept. The \( x \)-intercept is \( x = 5/3 \). Since that is the only value of \( x \) for which the numerator of \( r(x) \) is zero, there is only one \( x \)-intercept.

Example 2: Find the domain and intercepts of the rational function \( r(x) = \frac{x^2 - 4x - 2}{x^3 - 1} \).

Solution: To determine the domain we need to find those real numbers \( x \) for which \( x^3 - 1 = 0 \). The only solution to this equation is \( x = 1 \). Thus, the domain consists of all real numbers except \( x = 1 \).

The \( y \)-intercept is \( r(0) = \frac{-2}{-1} = 2 \). The \( x \)-intercepts are those values of \( x \) for which \( x^2 - 4x - 2 = 0 \). The solutions are \( x = 2 + \sqrt{6} \) and \( x = 2 - \sqrt{6} \). A plot of this rational function is shown below.

![Plot of the rational function](image-url)