Rational Expressions

Operations with Rational Expressions
The operations of addition, subtraction, multiplication, and division with rational expressions follow the same rules that are used with common fractions. The key, as with simplifying rational expressions, is to factor the polynomials.

I. Multiplication of Rational Expressions
Recall with fractions \( \frac{a}{b} \) and \( \frac{c}{d} \) that \( \frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd} \). Also recall when multiplying fractions that you may cross-cancel; that is, reduce common factors of any numerator with any denominator.

Example 1:
\[
\frac{2}{9} \cdot \frac{3}{10} = \frac{2 \cdot 3}{9 \cdot 10} = \frac{6}{90} = \frac{1}{15}
\]
The same is true of multiplying rational expressions. As before, the key is to factor the polynomials first.

Example 2: Simplify \( \frac{5}{x - 1} \cdot \frac{x^2 - 1}{25x - 50} \)
Solution: First factor the polynomials. Don’t forget to cancel common factors!
\[
\frac{5}{x - 1} \cdot \frac{x^2 - 1}{25x - 50} = \frac{5}{x - 1} \cdot \frac{(x + 1)(x - 1)}{25(x - 2)} = \frac{1}{1} \cdot \frac{x + 1}{5(x - 2)} = \frac{x + 1}{5x - 10}; \quad x \neq 1 \text{ or } 2
\]

Before looking at the next example, recall our strategies for factoring:
1. Factor out any common factors
2. If more than 3 terms, try factoring by grouping
3. Recognize special products
4. Factor trinomials using product/sum strategies:
   a) \( x^2 + bx + c \): factors into \( (x + m)(x + n) \) where \( mn = c \) and \( m + n = b \)
   b) \( ax^2 + bx + c \): split \( bx \) into \( mx + nx \), where \( mn = ac \) and \( m + n = b \), then factor by grouping.

Example 3: Simplify \( \frac{2x^2 + x - 6}{x^2 + 4x - 5} \cdot \frac{x^3 - 3x^2 + 2x}{4x^2 - 6x} \cdot \frac{x + 5}{x^3 - 8} \)
Solution:
\[
\frac{2x^2 + x - 6}{x^2 + 4x - 5} \cdot \frac{x^3 - 3x^2 + 2x}{4x^2 - 6x} \cdot \frac{x + 5}{x^3 - 8} = \frac{(2x - 3)(x + 2)}{(x + 5)(x - 1)} \cdot \frac{x(x - 2)(x - 1)}{2x(2x - 3)} \cdot \frac{x + 5}{(x - 2)(x^2 + 2x + 4)} = \frac{x + 2}{2(x^2 + 2x + 4)} = \frac{x + 2}{2x^2 + 4x + 8}; \quad x \neq 5, 1, 0, \frac{3}{2}, -2
Example 4: Simplify \( \frac{2x - 4}{x^2 - 4} \)

Solution:
\[
\frac{2x - 4}{x^2 - 4} = \frac{2(x - 2)}{(x - 2)(x + 2)} = \frac{2}{x + 2}, \quad x \neq 2.
\]

II. Division of Rational Expressions

Recall with fractions \( \frac{a}{b} \) and \( \frac{c}{d} \) that \( \frac{a}{b} \div \frac{c}{d} = \frac{ad}{bc} \). In other words, dividing by a fraction is the same as multiplying by its reciprocal. The same is true for rational expressions. As before, the key to making the work easier is to factor the polynomials first.

Example 5: \( \frac{x^2 + 2x - 15}{x + 2} \div (x^2 + 7x + 10) \)

Solution: Begin by writing the second term as \( \frac{1}{x + 2} \):

\[
\frac{x^2 + 2x - 15}{x + 2} \div \frac{x^2 + 7x + 10}{1} = \frac{x^2 + 2x - 15}{x + 2} \cdot \frac{1}{x^2 + 7x + 10}
\]

\[
= \frac{(x + 5)(x - 3)}{x + 2} \cdot \frac{1}{(x + 5)(x + 2)}
\]

\[
= \frac{x - 3}{(x + 2)^2} = \frac{x - 3}{x^2 + 4x + 4}, \quad x \neq -5, -2
\]

Notice that the \( x + 2 \) terms cannot be cancelled since both are in the denominator.

Multiplication and Division can also appear together. Only take the reciprocal of the fractions you are dividing.

Example 6: \( \left( \frac{x^2 - 7y + 12}{y^2 + 3y - 18} \right) + \frac{y^2 + 3y - 28}{y^2 + 12y + 36} \cdot \frac{3y + 21}{4y + 24} \)

Solution: We only take the reciprocal of the second fraction:

\[
\left( \frac{y^2 - 7y + 12}{y^2 + 3y - 18} + \frac{y^2 + 3y - 28}{y^2 + 12y + 36} \right) \cdot \frac{3y + 21}{4y + 24}
\]

\[
= \frac{y^2 - 7y + 12}{y^2 + 3y - 18} \cdot \frac{y^2 + 3y - 28}{y^2 + 12y + 36} \cdot \frac{3y + 21}{4y + 24}
\]

\[
= \frac{(y - 3)(y - 4)}{(y + 6)(y - 3)} \cdot \frac{(y + 6)^2}{(y + 7)(y - 4)} \cdot \frac{3(y + 7)}{4(y + 6)}
\]

\[
= 3; \quad y \neq -7, -6, 3, 4
\]

Question: Explain why the values \(-7, -6, 3, 4\) are excluded in the previous example.

Answer: The easiest way to understand why these values have been excluded is to write

\[
\left( \frac{y^2 - 7y + 12}{y^2 + 3y - 18} + \frac{y^2 + 3y - 28}{y^2 + 12y + 36} \right) \cdot \frac{3y + 21}{4y + 24}
\]

as a fraction.

\[
\frac{y^2 - 7y + 12}{y^2 + 3y - 18} \div \frac{3y + 21}{4y + 24}
\]

\[
\frac{y^2 + 3y - 18}{y^2 + 12y + 36} \div \frac{4y + 24}{4y + 24}
\]

\[
y \neq -6 \text{ follows since } 4y + 24 \text{ cannot equal } 0.
\]

We also get \( y \neq 3 \) because the term

\[
\frac{y^2 - 7y + 12}{y^2 + 3y - 18}
\]

\[
= \frac{y^2 - 7y + 12}{(y - 6)(y + 3)}
\]
The next observation is that the denominator of
\[
\frac{y^2 - 7y + 12}{y^2 + 3y - 18} - \frac{y^2 + 3y - 28}{y^2 + 12y + 36}
\]
that is, \(\frac{y^2 + 3y - 28}{y^2 + 12y + 36}\) cannot equal zero. Factoring the numerator and denominator of this rational expression we have
\[y^2 + 3y - 28 = (y + 7)(y - 4)\]
The numerator is zero if \(y = -7\) or if \(y = 4\). This accounts for excluding the numbers \(-7\) and 4. The denominator of
\[\frac{y^2 + 3y - 28}{y^2 + 12y + 36}\]
factors into \((y + 6)^2\) which means we have to exclude \(-6\), but we have already done that.

**Question:** What values of \(x\) are not allowed in the rational expression \(\frac{x - 1}{x - 6} + \frac{x - 3}{x - 4}\)?

**Answer:** 3, 4, 6 (The value 3 is not allowed because \(\frac{x - 3}{x - 4}\) is zero at \(x = 3\) and we cannot divide by 0. \(x = 4\) is not allowed because we have the term \(x - 4\) in the denominator. Similarly \(x = 6\) is not allowed because the term \(x - 6\) is the denominator of \(\frac{x - 1}{x - 6}\).)

**III. Addition and Subtraction of Rational Expressions**

Recall that addition and subtraction is done by first finding a common denominator. The least common denominator (LCD) of several fractions is the product of all prime factors of the denominators. A factor only occurs more than once in the LCD if it occurs more than once in any one fraction. Once you have the LCD, convert each fraction to an equivalent fraction with the LCD, then add or subtract the numerators.

**Example 7:** \(\frac{1}{x} - \frac{1}{x - 1}\)

**Solution:** The LCD here is \(x(x - 1)\). We convert each fraction to an equivalent one with this denominator: That is, \(\frac{1}{x} = \frac{x - 1}{x(x - 1)}\) and \(\frac{1}{x - 1} = \frac{x}{x(x - 1)}\)

\[
1 - 1 = \frac{1(x - 1)}{x(x - 1)} - \frac{1(x)}{(x - 1)(x)}
\]

\[
= \frac{x - 1}{x(x - 1)} - \frac{x}{x(x - 1)}
\]

\[
= \frac{(x - 1) - x}{x(x - 1)} = \frac{-1}{x(x - 1)}
\]

**Example 8:** \(\frac{y - 4}{y + 6} - \frac{y^2 + 3y - 28}{y^2 + 12y + 36}\)

**Solution:** We must factor the denominators first to find the LCD:
The LCD is \((y + 6)^2\). We now convert each fraction to an equivalent fraction using this denominator:

\[
\frac{y - 4}{y + 6} - \frac{y^2 + 3y - 28}{(y + 6)^2} = \frac{(y - 4)(y + 6)}{(y + 6)^2} - \frac{y^2 + 3y - 28}{(y + 6)^2} = \frac{y^2 + 2y - 24}{(y + 6)^2} - \frac{y^2 + 3y - 28}{(y + 6)^2} = \frac{y^2 + 2y - 24 - y^2 - 3y + 28}{(y + 6)^2} = \frac{-y + 4}{(y + 6)^2}
\]

Note that the minus sign in the numerator was distributed across the expression \(- (y^2 + 3y - 28) = -y^2 - 3y + 28\).

**Question:** What is the common denominator in

\[
\frac{x - 5}{(x - 7)(x + 1)} - \frac{1}{x^2 - 4}
\]

**Answer:** \((x - 7)(x + 1)(x^2 - 4)\)