Vectors

A Brief Description

As life becomes more complicated, entities arise that not only have a magnitude, but also a direction associated with them. Such objects are called vectors. A common example of this occurs when you give someone directions to get to your home: go 5 blocks west, then 16 blocks north, and my house is on the corner. There are two vectors here. The first has a magnitude of 5 units and a direction, west; the second has a magnitude of 16 and a direction, north. These verbal descriptions of a vector are too cumbersome, and a better notation has been invented. It is the same notation which is used to locate points in the plane, $\mathbb{R}^2$ or in three-space, $\mathbb{R}^3$.

Representing Vectors.

Example 1: A vector which is one unit long and points due north can be represented by the ordered pair $(1, 0)$, and a vector which has magnitude 5 and points northwest is $\left(-\frac{5}{\sqrt{2}}, \frac{5}{\sqrt{2}}\right)$.

The individual values are called components. Since these vectors have only two components, we think of them as belonging to or lying in a plane. We often picture vectors as arrows. The tail of the arrow is at the origin and the arrow tip is located at the point whose Cartesian coordinates are the same as the ordered pair (triple) which represents the vectors. The vectors $(1, 1)$, $(-4, 3)$, and $(3, -5)$ are plotted below.

Two dimensions

In $\mathbb{R}^2$ (the plane) there are two special vectors. The two that are parallel to the coordinate axes. They are commonly denoted by the symbols $\vec{i}$ and $\vec{j}$ or by the pair of symbols $\vec{e}_1$ and $\vec{e}_2$. That is,

$\vec{e}_1 = \vec{i} = (1, 0)$

$\vec{e}_2 = \vec{j} = (0, 1)$
Three dimensions
In $\mathbb{R}^3$ we have three special vectors which point in directions parallel to the coordinate axes.

$\vec{e}_1 = \vec{i} = (1, 0, 0)$

$\vec{e}_2 = \vec{j} = (0, 1, 0)$

$\vec{e}_3 = \vec{k} = (0, 0, 1)$

There is an ambiguity in this notation. What does $\vec{e}_2$ mean? Is it the ordered pair of numbers $(0, 1)$ or the ordered triple of numbers $(0, 1, 0)$. The answer is determined by the context of the particular discussion in which this symbol appears.

Coordinate vectors
These vectors $\{\vec{e}_1, \vec{e}_2\}$ are sometimes referred to as the coordinate vectors. That is, in $\mathbb{R}^2$ the standard coordinate vectors are $\vec{e}_1$ and $\vec{e}_2$.

**Question:** What are the coordinate vectors of $\mathbb{R}^3$?

**Answer:** The coordinate vectors in $\mathbb{R}^3$ are: $\vec{e}_1, \vec{e}_2, \text{ and } \vec{e}_3$.

**Example 2:** Suppose you tell someone the way to get to your house is to go 7 blocks east, then 1 block south, and finally 4 blocks east. One way to indicate this in vector notation is

$$(7, 0) + (0, -1) + (4, 0)$$

Or using the coordinate vectors

$$7\vec{e}_1 - \vec{e}_2 + 4\vec{e}_1 = 11\vec{e}_1 - \vec{e}_2$$

**Example 3:** Suppose you were given the following directions:

$$(1, 1) + (0, -4) + (3, 0) + (0, -2)$$.  

The English translation of the above would be go north-east, square root of 2 units, then south, 4 units, then east, three more units and finally south 2 units. Where would you be in the plane?

**Answer:** You would be 4 units east and 5 units south from the starting point. The coordinates are $(4, -5)$.

In vector notation: $(1, 1) + (0, -4) + (3, 0) + (0, -2) = (4, -5)$
Another use of vectors is to show a change of position. For example if something moves from point \( P = (a, b) \) to a point \( Q = (c, d) \), this change in position is commonly denoted by the vector \( \overrightarrow{PQ} = Q - P = (c - a, d - b) \).

**Example 4:** Suppose you trace a curve which starts at the point \((3, -4)\) and ends at the point \((5, 9)\). What vector describes the resultant of this motion?

**Solution:** Take the ending position and subtract the starting position from it. The desired vector is \((5, 9) - (3, -4) = (2, 13)\).

The following pictures give a graphic representation of this vector. The first is the actual vector; the second is the way to represent it. It is usually correct to represent any vector with originating point at the origin.

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**Example 5:** Represent the vector from \( P = (1, 3, 2) \) to \( Q = (4, 7, -1) \).

**Solution:** First compute the difference \((4, 7, -1) - (1, 3, 2) = (3, 4, -3)\). The picture is shown below.

Coordinates of the vector \((3, 4, -3)\)