Vectors

Vector Addition
The second algebraic operation is vector addition. This operation takes two vectors and adds them together to get a third vector. For example \((-2, 3) + (4, 1) = (2, 4)\). To get the components of the sum of these two vectors we add the corresponding components of the summands. Thus, \(-2 + 4 = 2\) and \(3 + 1 = 4\). This is the general rule as we state below

\[(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2).\]

Here the symbols \(x_1, x_2, y_1,\) and \(y_2\) represent arbitrary real numbers. This rule of addition is commonly referred to as the parallelogram law of addition. The reason for this is shown in the plot below where we sketch the sum of two vectors.

![Parallelogram Law of Addition](image_url)

The sum of \((-2, 3)\) and \((4, 1)\) is \((2, 4)\).

Notice that the sum of these two vectors is the diagonal of the parallelogram formed by the two summands.

**Example 1:** Find the sum of \((12, -1)\) and \((5, -2)\).

**Solution:** We have, adding the individual components, \((12, -1) + (5, -2) = (17, -3)\)

**Example 2:** Find the sum of \((6, -2)\) and \((-5, -2)\).

**Solution:** \((6, -2) + (-5, -2) = (1, -4)\)

Addition of vectors in any number of dimensions
Vector addition is defined similarly for two vectors in \(R^n\).

\[(x_1, x_2, \ldots, x_n) + (y_1, y_2, \ldots, y_n) = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n).\]

The reason for these particular definitions of scalar multiplication and vector addition is that vectors are used to model physical objects, and experience has demonstrated that these definitions accurately model how these objects combine with each other. Think back to your first physics course where forces, velocities, and accelerations were described by vectors. A simpler model surfaces when we remember how vectors are used to give directions on moving from one location to another.

Be aware that addition of two vectors is only defined if the vectors have the same number of components. Thus, a sum of the form \((2, 7) + (1, 1, 1)\) is not defined. The reason for not defining such an operation is that no one has seen a need for it.
Properties of the Algebraic Operations

We first list the algebraic properties which are possessed by the operations of vector addition and scalar multiplication. In the following, \( \vec{x}, \vec{y}, \) and \( \vec{z} \) represent arbitrary vectors in the same vector space. That is, all three vectors are in the same \( \mathbb{R}^n \); terms of the form \( \alpha \) and \( \beta \) will represent arbitrary scalars.

Let \( \vec{x}, \vec{y}, \) and \( \vec{z} \) represent arbitrary vectors and let \( \alpha \) and \( \beta \) be scalars. Then,

1. \( \vec{x} + \vec{y} = \vec{y} + \vec{x} \)

2. \( (\vec{x} + \vec{y}) + \vec{z} = \vec{x} + (\vec{y} + \vec{z}) \)

3. \( \alpha(\vec{x} + \vec{y}) = \alpha \vec{x} + \alpha \vec{y} \)

4. \( (\alpha + \beta)\vec{x} = \alpha \vec{x} + \beta \vec{x} \)

5. \( 1\vec{x} = \vec{x} \)

Proof of all five facts.

We will prove the truth of these equalities for vectors in \( \mathbb{R}^2 \). The proofs for the general case in \( \mathbb{R}^n \) are essentially the same. We let \( \vec{x} = (x_1, x_2) \), \( \vec{y} = (y_1, y_2) \) and \( \vec{z} = (z_1, z_2) \).

1. \( \vec{x} + \vec{y} = (x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2) \)
   \( = (y_1 + x_1, y_2 + x_2) = (y_1, y_2) + (x_1, x_2) = \vec{y} + \vec{x} \)

2. \( (\vec{x} + \vec{y}) + \vec{z} = ((x_1, x_2) + (y_1, y_2)) + (z_1, z_2) \)
   \( = (x_1 + y_1, x_2 + y_2) + (z_1, z_2) \)
   \( = ((x_1 + y_1) + z_1, (x_2 + y_2) + z_2) \)
   \( = (x_1 + (y_1 + z_1), x_2 + (y_2 + z_2)) \)
   \( = (x_1, x_2) + (y_1 + z_1, y_2 + z_2) \)
   \( = \vec{x} + (\vec{y} + \vec{z}) \)

3. \( \alpha(\vec{x} + \vec{y}) = \alpha((x_1, x_2) + (y_1, y_2)) = \alpha(x_1 + y_1, x_2 + y_2) \)
   \( = (\alpha x_1 + \alpha y_1, \alpha x_2 + \alpha y_2) \)
   \( = (ax_1 + \alpha y_1, ax_2 + \alpha y_2) \)
   \( = \alpha \vec{x} + \alpha \vec{y} \)

4. \( (\alpha + \beta)\vec{x} = (\alpha + \beta)(x_1, x_2) = ((\alpha + \beta)x_1, (\alpha + \beta)x_2) \)
   \( = (\alpha x_1 + \beta x_1, \alpha x_2 + \beta x_2) \)
   \( = (\alpha x_1 + \beta x_1, \alpha x_2 + \beta x_2) \)
   \( = \alpha \vec{x} + \beta \vec{x} \)

5. \( 1\vec{x} = 1(x_1, x_2) = (1x_1, 1x_2) = (x_1, x_2) = \vec{x} \)

Example 3: \( (2, -3, 7) + (1, 0, -5) = (3, -3, 2) = 3\vec{e}_1 - 3\vec{e}_2 + 2\vec{e}_3 \)

Example 4: \(-3(1, 2) + 5(-1, 4) = (-8, 14) = -8\vec{e}_1 + 14\vec{e}_2 \)
Example 5: Consider a plane which is flying at 25,000 feet due west with a speed of 500 miles per hour. Thus, the velocity of the plane can be represented by the vector \( \vec{V} = (-500, 0, 0) \). Suppose our plane suddenly experiences a down draft whose velocity is 125 miles per hour. Then, the resultant velocity of the plane is the sum of the two velocity vectors. That is, the resultant velocity equals \((-500, 0, 0) + (0, 0, -125) = (-500, 0, -125)\). The resultant speed of the plane is the length of this vector which is 
\[
\sqrt{(-500)^2 + 0^2 + (-125)^2} \approx 515.39 \text{ miles per hour}.
\]

Example 6: A swimmer sets out to cross a river 6 miles wide with a downstream current at 3 mi/h. He swims in a direction perpendicular to the bank at 2 mi/h. At what point downstream will the swimmer reach the other bank.

**Solution:** The net velocity of the swimmer is the vector \(3\vec{e}_1 + 2\vec{e}_2\). To cross the river the swimmer must travel 6 miles in the vertical direction. At a speed of 2 mi/h in the vertical direction, this will require 3 hours. Thus, the position vector after 3 hours will be \(3(3\vec{e}_1 + 2\vec{e}_2) = 9\vec{e}_1 + 6\vec{e}_2\), From this we see that the swimmer will be 9 miles downstream.

**Question:** What is the total distance travelled by the swimmer?

**Answer:** 
\[
\|9\vec{e}_1 + 6\vec{e}_2\| = \|(9, 6)\| = \sqrt{9^2 + 6^2} = \sqrt{81 + 36} = \sqrt{117} = 3\sqrt{13} \text{ miles}
\]