Vectors

Length

The length of any vector \((x_1, x_2)\) in \(R^2\) is defined to be the distance from the origin to the point which has the coordinates \((x_1, x_2)\). That is,

\[
\| (x_1, x_2) \| = \sqrt{x_1^2 + x_2^2}
\]

Thus, \(\| (5, -1) \| = \sqrt{5^2 + 1^2} = \sqrt{26}\).

A similar formula for the length of a vector is also defined for vectors in \(R^n\).

\[
\| (x_1, x_2, \ldots, x_n) \| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}.
\]

\(\| (2, -2, 4) \| = \sqrt{2^2 + (-2)^2 + 4^2} = 2\sqrt{6}\), and \(\| (1, 1, 1, 3) \| = \sqrt{1^2 + 1^2 + 1^2 + 3^2} = \sqrt{13}\).

The distance between two vectors \(\vec{x}\) and \(\vec{y}\) is defined to be the length of the vector \(\vec{x} - \vec{y}\). Since the length of a vector is the same as the length of the negative of the vector, it doesn’t matter if we compute \(\| \vec{x} - \vec{y} \|\) or \(\| \vec{y} - \vec{x} \|\). Thus, the distance between the vector \((2, 3)\) and the vector \((-5, 4)\) is

\(\| (2, 3) - (-5, 4) \| = \| (7, -1) \| = \sqrt{50}\).

Note: In case you were wondering why we use \(\| \|\) to denote the length of a vector, it is to remind ourselves that we are dealing with a vector and not a real number.

If we draw the triangle which has sides \((2, 3), (-5, 4)\), the third side is \((7, -1)\), and we see that the distance between these two vectors is just the length of the third side \((7, -1)\).
Properties

Let \( \mathbf{x} \) and \( \mathbf{y} \) represent arbitrary vectors and let \( \alpha \) and \( \beta \) be scalars. Then,

1. \[ \| \mathbf{x} \| = 0 \text{ if and only if } \mathbf{x} \text{ is the zero vector.} \]
2. \[ \| \alpha \mathbf{x} \| = |\alpha| \| \mathbf{x} \| \text{ That is, the length of a rescaled vector is the same (in magnitude) rescaling of the length.} \]
3. \[ \| \mathbf{x} + \mathbf{y} \| \leq \| \mathbf{x} \| + \| \mathbf{y} \| \text{ This inequality is called the triangle inequality. The reason for this is that if one labels the lengths of the sides of a triangle as } \| \mathbf{x} \| \text{ and } \| \mathbf{y} \|, \text{ then the third side has length } \| \mathbf{x} + \mathbf{y} \|. \text{ The inequality is just another way of saying the shortest distance between two points is the straight line between them.} \]

Proof: These statements will be verified for vectors in \( \mathbb{R}^2 \). The proofs for \( \mathbb{R}^3 \) and \( \mathbb{R}^n \) are somewhat similar.

1. If \( \| \mathbf{x} \| = 0 \), where \( \mathbf{x} = (x_1, x_2) \), then \( x_1^2 + x_2^2 = 0 \). Since the sum of two nonnegative numbers (\( x_1^2 \) and \( x_2^2 \)) can equal zero if and only if each of them is zero, we must have \( x_1^2 = 0 \) and \( x_2^2 = 0 \). Hence \( \mathbf{x} = (0, 0) = \mathbf{0} \). Conversely if \( \mathbf{x} = (0, 0) \), then \( \| \mathbf{x} \| = \sqrt{0 + 0} = 0 \).

2. \[ \| \alpha \mathbf{x} \| = \|\alpha(x_1, x_2)\| = \|\alpha x_1, \alpha x_2\| \]
   \[ = \sqrt{(\alpha x_1)^2 + (\alpha x_2)^2} \]
   \[ = \sqrt{\alpha^2 (x_1^2 + (x_2)^2)} \]
   \[ = \sqrt{\alpha^2} \sqrt{x_1^2 + (x_2)^2} = |\alpha| \| \mathbf{x} \| \]

3. Before verifying this inequality we need a few preliminary inequalities. The first one, \( 0 \leq (x_2y_1 - x_1y_2)^2 \) is obvious and is used to verify the inequality \( x_1y_1 + x_2y_2 \leq \sqrt{(x_1)^2 + (x_2)^2} \sqrt{(y_1)^2 + (y_2)^2} \).

This last inequality looks formidable, but it follows from the first inequality. (Square both sides of the second inequality and simplify.). We are now ready to verify the triangle inequality.

\[ \| \mathbf{x} + \mathbf{y} \|^2 = (x_1 + y_1)^2 + (x_2 + y_2)^2 \]
\[ = (x_1^2 + 2x_1y_1 + y_1^2) + (x_2^2 + 2x_2y_2 + y_2^2) \]
\[ = x_1^2 + x_2^2 + 2(x_1y_1 + x_2y_2) + y_1^2 + y_2^2 \]
\[ \leq x_1^2 + x_2^2 + 2\sqrt{(x_1)^2 + (x_2)^2}\sqrt{(y_1)^2 + (y_2)^2} + y_1^2 + y_2^2 \]
\[ = (\sqrt{x_1^2 + x_2^2} + \sqrt{y_1^2 + y_2^2})^2 = \| \mathbf{x} + \mathbf{y} \|^2 \]

Thus, we have shown (take the square root of both sides) that the triangle inequality \( \| \mathbf{x} + \mathbf{y} \| \leq \| \mathbf{x} \| + \| \mathbf{y} \| \) is valid.
Since the sum of two vectors is the vector equal to the sum of the corresponding components, we can use this fact to resolve vectors into horizontal and vertical components as in the following example.

**Example 1:** A weight of 50 pounds hangs from two wires as shown in the figure below. Find the tensions in each wire and their magnitudes.

The vectors \( \mathbf{T}_1 \) and \( \mathbf{T}_2 \) satisfy the equation \( \mathbf{T}_1 + \mathbf{T}_2 = (0, 50) \), since the downward force of the weight must be balanced by an equal and opposite force. \( \mathbf{T}_2 = (-T_2 \cos 50, T_2 \sin 50) \), where \( T_2 \) is the magnitude or length of the vector \( \mathbf{T}_2 \).

**Question:** Why is there a minus sign in the first component of \( \mathbf{T}_2 \)?

**Answer:** Because the vector \( \mathbf{T}_2 \) points in the negative \( x \) direction, so its \( x \) component must be negative.

Similarly \( \mathbf{T}_1 = (T_1 \cos 45, T_1 \sin 45) \). This leads to the equations

\[
T_1 \cos 45 - T_2 \cos 50 = 0 \\
T_1 \sin 45 + T_2 \sin 50 = 50
\]

Solving the second equation for \( T_1 \) and substituting this expression into the first equation, we get that \( T_2 \approx 35.49 \) which then gives \( T_1 \approx 32.26 \). Since we know the magnitudes of the vectors and their directions, we know the vectors.

\[
\mathbf{T}_1 \approx (-32.26 \cos 45, 32.26 \sin 45) \\
\mathbf{T}_2 \approx (35.49 \cos 50, 35.49 \sin 50)
\]