Vectors

Equation of a Plane

A plane in $\mathbb{R}^3$ is characterized by specifying a point $x_0$ on the plane, and a direction $\vec{N}$. The plane is all points $x$ for which the vector from the given point to this point, $x - x_0$, is perpendicular to $\vec{N}$.

Example 1: We want an equation for the plane which passes through the point $(-5, 6, -1)$ and which is perpendicular to the vector (direction) $\vec{p} = (3, 5, 7)$.

Solution: Let $\vec{r} = (x_1, x_2, x_3)$ be any other point on the plane. Then the vector $\vec{r} - (-5, 6, -1) = (x_1, x_2, x_3) - (-5, 6, -1) = (x_1 + 5, x_2 - 6, x_3 + 1)$ lies in the plane and hence must be perpendicular to the vector $\vec{p}$. This, of course, means that the dot product of these two vectors must be zero. This fact gives the desired equation.

$$(x_1 + 5, x_2 - 6, x_3 + 1) \cdot (3, 5, 7) = 3(x_1 + 5) + 5(x_2 - 6) + 7(x_3 + 1) = 3x_1 + 5x_2 + 7x_3 - 8 = 0.$$ 

Thus, an equation for this plane is $3x_1 + 5x_2 + 7x_3 = 8$.

Pay particular attention to the coefficients of the variables $x_i$. They are the components of the vector $\vec{p}$. A vector which is normal to the plane. The constant 8 depends upon which point the plane passes through. Note, this constant must be zero for any plane which passes through the origin.

Example 2: Find the equation of the plane which passes through the point $(13, 4, -6)$ and is perpendicular to the vector $(2, 7, 11)$.

Solution: Let, as we did above, $(x_1, x_2, x_3)$ be an arbitrary point on the plane. Then we must have

$$0 = ((x_1, x_2, x_3) - (13, 4, -6)) \cdot (2, 7, 11)$$
$$0 = (x_1 - 13, x_2 - 4, x_3 + 6) \cdot (2, 7, 11)$$
$$0 = 2(x_1 - 13) + 7(x_2 - 4) + 11(x_3 + 6) = 2x_1 + 12 + 7x_2 + 11x_3$$

or

$$-12 = 2x_1 + 7x_2 + 11x_3$$

Just to check our work, we pick one particular point on the plane, and verify that the direction from it to the given point on the plane is really perpendicular to the given direction. To get one point on the plane set $x_2$ and $x_3$ equal to zero in the equation and solve for $x_1$. This gives us the point $(-6, 0, 0)$; the vector $(-6, 0, 0) - (13, 4, -6) = (-19, -4, 6)$ must then lie in the plane, and the dot product of $(-19, -4, 6)$ with the vector $(2, 7, 11)$ is zero, which substantiates our claim that this equation does indeed describe the desired plane.
Example 3: Find an equation of the plane which passes through the three non-colinear points \((-1, 1, -1), (0, 2, -5), \) and \((5, 5, -4)\).

Solution: If we knew a direction which was perpendicular to the plane we could repeat the computations which we saw in the previous section. What we can calculate are two vectors which lie in the plane. If we then take their cross product, we will have a vector which is perpendicular to both of the vectors and hence to the plane they lie in.

Two pairs of non-parallel vectors which lie in the plane are:

\[
(0, 2, -5) - (-1, 1, -1) = (1, 1, -4)
\]

and

\[
(5, 5, -4) - (-1, 1, -1) = (6, 4, -3)
\]

The cross product of these two vectors is \((1, 1, -4) \times (6, 4, -3) = (13, -21, -2)\).

Thus, an equation for the plane containing the given points is:

\[
((x_1, x_2, x_3) - (-1, 1, -1)) \cdot (13, -21, -2) = 0
\]

\[
((x_1 + 1, x_2 - 1, x_3 + 1)) \cdot (13, -21, -2) = 0
\]

\[
13(x_1 + 1) - 21(x_2 - 1) - 2(x_3 + 1) = 0
\]

\[
13x_1 + 32 - 21x_2 - 2x_3 = 0
\]

\[
13x_1 - 21x_2 - 2x_3 = -32
\]

Verify: If we let \(f(x_1, x_2, x_3) = 13x_1 - 21x_2 - 2x_3\), then to verify we have found an equation for the plane containing the three given points, we verify that the function \(f\) takes on the value \(-32\) at each of the three points.

\[
f(-1, 1, -1) = -32
\]

\[
f(0, 2, -5) = -32
\]

\[
f(5, 5, -4) = -32
\]