Vectors

Exercises

1. Find a vector which represents moving from the point \( P \) to the point \( Q \) for each of the following pairs of points. For a - d, sketch the vector.
   a. \( P = (1, 1), Q = (5, -3) \)
   b. \( P = (-3, 7), Q = (-8, -12) \)
   c. \( P = (-2, 7, 4), Q = (14, 6, 10) \)
   d. \( P = (19, 5, 23), Q = (13, -4, 17) \)
   e. \( P = (3, 0, -6, 3), Q = (-4, 4, -5, 0) \)

2. A dog runs 4.5 miles in a direction 35° degrees north of east. Assuming that the origin is the dog’s starting point and that east is the positive \( x \) axis, what are the coordinates of the dog’s location?

3. A flea is on a cat. If the cat goes 5 feet in a direction 40° degrees (measured counter clockwise from the positive \( x \) axis) and the flea then gets off the cat and goes 1 foot in a direction of 90° degrees with respect to the positive \( x \) axis, what are the locations of the flea and the cat?

4. If \( \vec{x} = (2, -5) \), what angle does \( \vec{x} \) make with the positive \( x \) axis?

5. If an ant starts at the point \( (1, 2) \) and stops at the point \( (-4, 5) \), what vector represents the change of position of the ant?

6. If a vector starts at the point \( (3, -7) \) and terminates at the point \( (14, 5) \), what is the tangent of the angle this vector makes with the positive \( x \) axis?

7. If a woman was planing on walking 2 miles in a northern direction, but a companion persuades her to walk twice as far, how far will she walk and in which direction?

8. If the product of a vector \( \vec{x} \) with the number 3 is equal to \( (4, 15) \), what is \( \vec{x} \)?

9. Compute the following scalar products:
   a. \( 5(6, 13) \)
   b. \(-3(13, 4, -7) \)
   c. \( 2(3, 12, 4, -5, 8) \)

10. Does the equation \( a(3, 4) = (-15, -20) \) have a solution? If yes, find it.

11. Does the equation \( a(3, 4) = (-15, 20) \) have a solution? If yes, find it.

12. For each of the following, mentally compute the answer, then check the answer.
   a. \( 2(3, -4, 5, 1) \)
   b. \(-1(1, 2, 3, -5) \)
   c. \( 7(23, -2) \)

13. For each of the following draw the vector, and then draw the result of the scalar product.
   a. \( 2(-1, -1) \)
   b. \(-3(-1, 2) \)

14. For each of the vectors below draw the first vector with base at the origin and the second with its base at the point (2, 0). Note that the second vector is a scalar multiple of the first, and that the two vectors are parallel.
   a. \( (-1, -2) \) and \(-2(-1, -2) \)
   b. \( (1, 1) \) and \( 3(1, 1) \)

15. Rewrite each of the following vectors in terms of the \( \vec{i}, \vec{j}, \) and \( \vec{k} \) vectors.
   a. \( 2(3, -7) - 5(13, 5) \)
   b. \( (1, 3) + 5(-2, 1) - 7(6, 5) \)
   c. \( (1, 2, 0) + (1, 0, -4) \)
   d. \( -3(3, -1, 5) + 2(5, 5, 3) \)

16. Sketch the set of vectors which have the form \( t(1, 1) \) for all values of \( t \) which lie between -2 and 1.

17. Find all vectors \( \vec{x} \) which satisfy the vector equation \( 2\vec{x} + 5(7, 8) = (-3, 7) \).
18. Find all vectors \( \vec{x} \) which satisfy the vector equation \( 2\vec{x} + 5(1, -2, 4) = (5, 12, 17) \).

19. Sketch the set of vectors \( s(1, 2) + t(0, 1) \) for all values of \( s \) and \( t \) which satisfy the constraints \( 0 \leq s \leq 1 \) and \( -1 \leq t \leq 0 \).

20. Is there a scalar \( a \) such that \( a(1, 1) = (1, 2) \)?

21. Are there scalars \( a \) and \( \beta \) such that \( a(1, 2) + \beta(-2, 3) = (7, 9) \)?

22. For each of the following, mentally compute the answer, then check your answer.
   a. \( (3, -4, 5, 1) + (1, 12, -4, 5) \)
   b. \( (1, 2) - (7, 15) \)
   c. \( (23, -2) + (-5, 11) \)

23. For each of the following draw the individual vectors, and then draw the result of the vector addition.
   a. \( (1, -1) + (2, 5) \)
   b. \( (1, 2) - (-2, -3) \)

24. For each of the following pairs \( \vec{x} \) and \( \vec{y} \) compute \( \vec{x} + \vec{y}, \vec{x} - \vec{y}, \) and \( 2\vec{x} + 3\vec{y} \).
   a. \( \vec{x} = (1, 2), \vec{y} = (-1, 2) \)
   b. \( \vec{x} = (-3, 2, 5), \vec{y} = (5, 1, -4) \)
   c. \( \vec{x} = (0, 1, 3, -7), \vec{y} = (12, -3, 2, 9) \)

25. If a vector \( \vec{x} \) is written as the sum of scalar multiples of vectors \( \vec{a} \) and \( \vec{b} \), i.e., \( \vec{x} = a\vec{a} + b\vec{b} \), we say that \( \vec{x} \) is a linear combination of \( \vec{a} \) and \( \vec{b} \). For each of the following pairs of vectors \( \vec{a} \) and \( \vec{b} \), write the vector \( \vec{t} = (1, 0) \) as a linear combination of \( \vec{a} \) and \( \vec{b} \).
   a. \( (1, 1) \) and \( (0, 1) \)
   b. \( (1, 1) \) and \( (-1, 1) \)
   c. \( (2, 5) \) and \( (-5, 2) \)

26. Compute the lengths of the following vectors.
   a. \( (2, 3) \)
   b. \( -4(-3, 7) \)
   c. \( 7(3, 3) + 5(-1, 8) \)
   d. \( (2, 3, -5) \)

27. If a dog runs from the point with coordinates \((-2, 6)\) to the point \((5, -1)\), how far has the dog run? If it takes the dog four hours to run between the two points, how fast is the dog running. Assume that the units in the coordinate system are miles.

28. An airplane flies from the point \((1, 1, 3)\) to the point \((5, -5, 0, 1)\). Once again the unit distance is one mile. If it takes the plane 4 minutes to make this descent, how fast is the plane descending in miles per hour?

29. Find a unit vector which is parallel to the vector which points from the point \((5, 13)\) to \((17, 11)\).

30. A flea is walking on a coordinate plane. If the flea starts at the point \((1, 0)\), and then hops to each of the points \((2, 1), (3, 0), (1, -2), \) and finally \((-1, -3)\) in succession, how far has the flea hopped, and what vector points from the flea’s initial position to its final position?

31. You are flying an airplane at an altitude of 5000 feet and heading in a north-east direction. The control tower tells you to descend to an altitude of 1000 feet while simultaneously heading straight south. Find a unit vector which points in the direction you should head the plane.

32. For each pair of vectors \( \vec{x} \) and \( \vec{y} \) compute their lengths and the lengths of \( \vec{x} + \vec{y} \) and \( \vec{x} - \vec{y} \). Then sketch the four vectors.
   a. \( \vec{x} = (1, 1) \) and \( \vec{y} = (1, -1) \)
   b. \( \vec{x} = (1, 1) \) and \( \vec{y} = (1, 0) \)
   c. \( \vec{x} = (2, -1) \) and \( \vec{y} = (1, 2) \)
   d. \( \vec{x} = (1, 0) \) and \( \vec{y} = (0, -1) \)

33. For each of the vectors given, find a unit vector which points in the same direction as the vector.
   a. \( (1, 2) \)
   b. \( (3, 5) \)
   c. \( (2, 3, -5) \)
34. Two butterflies are pulling a sled which resists motion with a force of 2000 pounds. Assume that each butterfly is exerting the same force and that each of them is pulling at a 5° angle to the direction of motion. How much force is each butterfly exerting?

35. Compute the dot product for the following pairs of vectors.
   a. \((1, 2), (3, -7)\)
   b. \((2, 14, 5), (-2, 0, 7)\)
   c. \(((1, -2) + (-3, -8)), (3, 2)\)
   d. \((3, 4, -3, 1), (5, -3, 0, 6)\)

36. Find all unit vectors that are perpendicular to the vector \((2, -3)\).

37. A man wants to walk in a direction which is perpendicular to the vector \((7, 13)\) and for which the \(x\) coordinate is decreasing. Find a unit vector which points in this direction.

38. Find the angle between the two vectors \((1, 3)\) and \((4, -2)\).

39. Find the angle between the two vectors \((2, 3)\) and \((-5, 7)\).

40. Find the angle between the following pairs of vectors
   a. \((1, 1), (9, -4)\)
   b. \((2, 3, -8), (2, 2, 13)\)
   c. \((1, 2, -3, 7), (-1, 17, 5, 2)\)

41. Find all vectors which make an angle of 45 degrees with the vector \((1, 1)\) and have length 1.

42. Let \(\vec{X} = (3, 17), \vec{Y} = (4, -1)\). Is the vector \(\vec{N} = (0.5, -0.125)\) the projection of \(\vec{X}\) onto \(\vec{Y}\)?

43. Find the projection of \((-1, 1, -1)\) onto \((2, 0, 5)\).

44. Sketch the following vectors \((2, -3), (1, 2)\), the projection of \((2, -3)\) onto \((1, 2)\), and finally the projection of \((1, 2)\) onto \((2, -3)\).

45. Determine if the following pairs of vectors are perpendicular.
   a. \((1, 2), (3, 3)\)
   b. \((2, 1, 3), (0, 3, -1)\)
   c. \((4, 3, 0, -10, 2), (1, -1, 5, -6, 2)\)

46. Find two vectors whose sum is the vector \((11, 7)\) such that one of the vectors is parallel to the vector \((-1, 1)\) and the second is perpendicular to the vector \((-1, 1)\).

47. A force is applied to a sled. The force is modeled by the vector \((1, 15, -1)\). Resolve this force into two components, one perpendicular to the \(x, y\) plane and the second parallel to the \(x, y\) plane. The second component, the one parallel to the \(x, y\) plane is the force which will cause the sled to move. What is the magnitude, i.e., length, of this component.

48. Suppose the vertices of a triangle are \(A(1, 1), B(2, 5), C(4, 3)\). What are the three angles of this triangle?

49. A man is walking and pulling a coaster wagon. The angle of the wagon’s handle makes a 45° angle with the ground. Suppose the man exerts a force of 50 pounds along the handle. What is the component of force parallel to the ground, and how much work is being done if the wagon is moved 100 feet? Use the diagram below

Resolve the force into two perpendicular forces: use the component in the direction of motion.
50. Suppose a 50 pound object is sliding down an inclined ramp and the ramp is inclined at a 25° angle to the ground. If the ramp is 20 feet long, how much work does a man do, if he is pushing the object up against the object with just enough force to counteract the weight of the object as it slides down the ramp? Hint. Use the diagram below.

![Diagram of a ramp with weight and forces](image)

51. Compute the cross product of \((-4, 5, 6)\) and \((1, 2, -8)\).

52. Compute the cross product of \((0, -2, 3)\) and \((0, 5, 4)\).

53. Compute the cross product of the \((2, 0, 7)\) and \((6, 0, -1)\).

54. If \(\vec{X}\) and \(\vec{Y}\) are two nonparallel vectors, then the length of their cross product is \(\|\vec{X}\| \|\vec{Y}\| \sin \theta\)

where \(\theta\) is the smaller of the two angles determined by the two vectors. Show that if \(\vec{X}\) and \(\vec{Y}\) are two vectors in the \(x, y\) plane, then the length of their cross product equals the area of the parallelogram determined by the two vectors.

The result of this problem is valid for any two vectors in \(R^3\).

55. Without using the formula for the cross product of two vectors, determine \((2, 0, 0) \times (0, -3, 0)\).

56. Compute that area of the parallelogram formed by the two vectors \((3, 2, 8)\) and \((1, 2, -3)\).

57. Find two equations for the straight line joining the points \((1, 1)\) and \((2, -3)\).

58. Find an equation for the straight line joining the two points \((2, -5, 6)\) and \((9, -3, 4)\).

59. Does the point \((-1, 1, -1)\) lie on the straight line containing the two points \((1, 1, 1)\) and \((-3, 4, 5)\)?

60. Find an equation for the straight line which is parallel to the vector \((2, -5)\) and passes through the point \((-4, 7)\).

61. Do the straight lines \((1, 1) + t(5, 9)\) and \((3, -2) + t(6, 7)\) intersect?

62. Do the straight lines \((-1, 3, -6) + t(1, 1, -2)\) and \((1, 1, -1) + t(-1, 7, 4)\) intersect?

63. Does the point \((3, 2, 6)\) lie on the line \((3, 4, -1) + t(1, -4, 5)\)?

64. Find the angles made by the intersections of the diagonals of a rectangle with sides of lengths 4 and 10.

65. Find two non-parallel vectors which are parallel to the plane \(3x - 2y + z = 1\).

66. Find an equation for the plane which passes through the point \((1, 2, -3)\) and is perpendicular to the vector \((-3, 4, 6)\).

67. Find two points which lie on the plane \(-x + 5y + 4z = 5\).

68. If the points \((1, 1, 1)\), \((2, -3, 0)\) and \((-3, 4, 1)\) lie on a plane, find a vector which is perpendicular to this plane.

69. Find an equation of the plane which contains the line \((3, 4, -1) + t(1, -4, 5)\) and the point \((3, 2, 6)\).

70. Find the point of intersection of the plane \(2x + y + 3z = 1\) and the line \(\frac{x}{2} = \frac{y + 3}{5} = \frac{z - 5}{7}\).

71. Use vector projections to show that the distance from a point \(P(x, y, z)\) to the plane \(ax + by + cz = d\) is \(\frac{|ax + by + cz - d|}{\sqrt{a^2 + b^2 + c^2}}\). Use this formula to find the distance from the point \((1, -2, 1)\) to the plane \(3x - y + z = 2\).

72. Find the distance from the point \((3, 4)\) to the line \(y = -2x + 1\).

73. Sketch each of the following parametric curves by hand, and then check with your computer.
   a. \((\cos t, 2 \sin t)\)
   b. \((t^2, t)\)
c. \((t - 3, t + 2)\)
d. \((\cos^2 t, \cos^4 t)\)
e. \((\sin t, \cos^2 t)\)
f. \((\frac{1 - t}{1 + t}, \frac{t}{1 + t})\)
g. \((3 + \cos t, -2 + \sin t)\)
h. \((1 + 2\sin t, 3\cos t)\)

74. A particle is moving in the plane and its path is given by the parametric equations \(x(t) = 2t + 1\) and \(y(t) = -\frac{t^2}{16} + 5t\). Where is the particle at \(t = 0\)? At what values of \(t\) will the particle lie on the \(x\) axis? What are the \(x\) coordinates of the particle when it is on the \(x\) axis? Find a parametric curve which traces out the circle of radius 2 centered at the point \((1, -2)\).

75. Find a parametric curve which models a particle traveling along the \(x\)-axis from \(x = 0\) to \(x = 1\) and then back to \(x = 0\).

76. Find a parametric curve which looks like the letter C

77. Find a parametric curve which traces out the straight line paths joining the three points \((1, 1, -3)\), \((-2, 5, 1)\), and \((7, -4, 5)\) in the given order

78. Find a parametric curve which traces out the ellipse \(\frac{x^2}{4} + \frac{y^2}{25} = 1\)

79. Find a parametric curve which traces out the graph of \(y = x^2\) from the point \((2, 4)\) to the origin