Sequences and Series

Introduction to Series

Definitions:
- A **series** is a sequence of terms that you intend to add up.
- A **finite series** has a finite number of terms. The notation is given below:
  \[ \sum_{n=1}^{k} a_n = a_1 + a_2 + a_3 + \cdots + a_k \]
- An **infinite series** has an infinite number of terms. We write the series as:
  \[ \sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots \]
  The initial index may or may not be 1.
- Given an infinite series \( S = \sum_{n=1}^{\infty} a_n \), its \( k \)-th partial sum is the finite series \( S_k = \sum_{n=1}^{k} a_n \). Then, the sum of the infinite series is the limit of the sequence of partial sums, \( S_1, S_2, \ldots, S_n, \ldots \), provided the limit exists or is positive or negative infinity. In symbols, we write:
  \[ S = \lim_{k \to \infty} S_k \quad \text{or} \quad \sum_{n=1}^{\infty} a_n = \lim_{k \to \infty} \sum_{n=1}^{k} a_n \]

Thus, an infinite series is associated with two sequences:
- the sequence of terms: \( \{a_n\} \)
- the sequence of partial sums: \( \{S_k\} \).

The sum of the series (or simply the series) is the sum of the sequence of terms and is the limit of the sequence of partial sums.

Further Terminology:
- If the limit exists (i.e. \( S = \lim_{k \to \infty} S_k \) is finite), we say the series exists or is convergent or converges to \( S \) or has sum \( S \).
- If the limit does not exist (i.e. \( \lim_{k \to \infty} S_k \) does not exist), we say the series does not exist or is divergent or diverges or does not have a sum.
- If the limit is positive infinity (i.e. \( \lim_{k \to \infty} S_k = \infty \)), we say the series diverges to \( \infty \).
- If the limit is negative infinity (i.e. \( \lim_{k \to \infty} S_k = -\infty \)), we say the series diverges to \( -\infty \).

To say that the limit is positive or negative infinity does **not** say that the limit exists! It merely says the way in which it does not exist, i.e. the way in which it **diverges**.

Example 1: Find the sum of the series \( \sum_{n=1}^{\infty} (-1)^{n+1} = 1 - 1 + 1 - 1 + \cdots \).

Solution: The \( k \)-th partial sum is
\[
S_k = \sum_{n=1}^{k} (-1)^{n+1} = 1 - 1 + 1 - 1 + \cdots + (-1)^{k+1}
\]
\[
= \begin{cases} 
  1 & \text{if } k \text{ is odd} \\
  0 & \text{if } k \text{ is even}
\end{cases}
\]

Since the partial sums alternate between 1 and 0 forever, the limit of the partial sums does not exist and the series does not have a sum; the series is divergent.