Solving Quadratic Equations

A quadratic equation is an equation that can be written in the form

\[ ax^2 + bx + c = 0 \]

This is referred to as the standard form of a quadratic equation. We will discuss three methods for solving quadratic equations: factoring, completing the square, and using the quadratic formula.

I. Solving by Factoring

The following theorem is the basis for solving equations by factoring:

**Theorem:** If \( ab = 0 \), then \( a = 0 \) or \( b = 0 \) or both.

Therefore, the following strategy can be applied to solve equations by factoring:

1. Write the equation in standard form (if it is not already in standard form)
2. Factor
3. Set each factor equal to zero and solve.

Example 1: Solve \( x^2 = 5x - 6 \) for \( x \)

Solution: We first write the equation in standard form:

\[
\begin{align*}
x^2 &= 5x - 6 & \text{subtract 5x from both sides} \\
x^2 - 5x &= 5x - 5x - 6 \\
x^2 - 5x &= -6 & \text{add 6 to both sides} \\
x^2 - 5x + 6 &= -6 + 6 \\
x^2 - 5x + 6 &= 0 & \text{now factor the left} \\
(x - 2)(x - 3) &= 0 & \text{set each factor equal to 0 and solve} \\
x - 2 &= 0 \text{ or } x - 3 = 0 \\
x &= 2 \text{ or } x = 3
\end{align*}
\]

We can check both of our solutions:

\[
\begin{align*}
2^2 &= 5(2) - 6? & 3^2 &= 5(3) - 6? \\
4 &= 10 - 6? & 9 &= 15 - 6? \\
4 &= 4\sqrt{ } & 9 &= 9\sqrt{ }
\end{align*}
\]

Both 2 and 3 are solutions.
Example 2: Solve \(2a^2 + 3a - 14 = 0\) for \(a\)

Solution: The equation is already in standard form, so factor and solve:

\[
2a^2 + 3a - 14 = 0 \\
(2a + 7)(a - 2) = 0 \\
2a + 7 = 0 \text{ or } a - 2 = 0 \\
2a = -7 \text{ or } a = 2 \\
a = -\frac{7}{2} \text{ or } a = 2
\]

We can check both of our solutions:

\[
2\left(-\frac{7}{2}\right)^2 + 3\left(-\frac{7}{2}\right) - 14 = 0 \quad 2(2)^2 + 3(2) - 14 = 0 \\
2\left(\frac{49}{4}\right) + 3\left(-\frac{7}{2}\right) - 14 = 0 \quad 2(4) + 3(2) - 14 = 0 \\
\frac{49}{2} - \frac{21}{2} - 14 = 0 \quad 8 + 6 - 14 = 0 \\
\frac{49}{2} - \frac{21}{2} - \frac{28}{2} = 0 \quad 0 = 0 \checkmark
\]

Therefore, both \(-\frac{7}{2}\) and 2 are solutions to the equation.

II. Solving by Completing the Square

The equation \(x^2 = 9\) can easily be solved by knowing that \(3^2 = 9\) (so \(x\) must be 3). However, there is another number whose square is 9. Recall that the product of two negative numbers is a positive number. Thus \(x = -3\) is also a solution to the equation since \((-3)^2 = (-3)(-3) = 9\).

In general, the solution to the equation \(x^2 = c\) (where \(c\) is positive) is \(x = \sqrt{c}\) or \(x = -\sqrt{c}\). This can be written more concisely as \(x = \pm \sqrt{c}\). This method is often referred to as the square root extraction method or simply the square root method.

Example 3: Solve each of the following equations.

a) \(x^2 = 40\)  
b) \((x - 2)^2 = 17\)

Solution:  
a) The solution is given by (remember to simplify your radicals!)  
\[
x = \pm \sqrt{40} \\
x = \pm 2\sqrt{10}
\]

b) Note that if  
\((x - 2)^2 = 17\)
then  
\[
x - 2 = \pm \sqrt{17}
\]
or (adding 2 to both sides)  
\[
x = 2 \pm \sqrt{17}
\]
Given a quadratic equation $ax^2 + bx + c = 0$ (especially one which is not easily factored), we can rewrite the equation so that the left-hand side is a perfect square (like the second example above). The process, which works on every quadratic expression, is described below:

**Example 4:** Solve $y^2 + 7y + 3 = 0$ by completing the square

Solution 1: We begin by arranging the left-hand side to look like $y^2 + ky$.

\[
y^2 + 7y + 3 = 0 \quad \text{subtract 3 from both sides}
\]
\[
y^2 + 7y = -3
\]

Now we half and square the linear coefficient (the number in front of $y$) and add this number to BOTH sides of the equation:

\[
y^2 + 7y = -3 \quad \text{add } \left(\frac{7}{2}\right)^2 = \frac{49}{4}
\]
\[
y^2 + 7y + \frac{49}{4} = -3 + \frac{49}{4}
\]
\[
y^2 + 7y + \frac{49}{4} = \frac{-12}{4} + \frac{49}{4}
\]
\[
y^2 + 7y + \frac{49}{4} = \frac{37}{4}
\]

Why did we do this? Because the left-hand side is now a perfect square. Recall the special product

\[(x + a)^2 = x^2 + 2ax + a^2\]

Notice that if we take one-half the coefficient of $x$ and square it, we obtain

\[\left(\frac{2a}{2}\right)^2 = a^2\]

This will work no matter what $a$ is.

Since the left hand side is a perfect square we can "undo" the square by taking the square root of both sides.

\[
y^2 + 7y + \frac{49}{4} = \frac{37}{4}
\]
\[
\left(y + \frac{7}{2}\right)^2 = \frac{37}{4}
\]
\[
y + \frac{7}{2} = \pm \sqrt{\frac{37}{4}}
\]
\[
y + \frac{7}{2} = \pm \frac{\sqrt{37}}{2}
\]
\[
y = -\frac{7}{2} \pm \frac{\sqrt{37}}{2} = \frac{-7 \pm \sqrt{37}}{2}
\]

final solution

Solution 2: Later it will be more useful when completing the square to organize our numbers as follows:

\[
y^2 + 7y + 3 = 0
\]
\[
(y^2 + 7y) + 3 = 0
\]
\[
\left(y^2 + 7y + \left(\frac{7}{2}\right)^2\right) + 3 - \left(\frac{7}{2}\right)^2 = 0
\]
\[
\left(y + \frac{7}{2}\right)^2 - \frac{37}{4} = 0
\]

From here, we add $\frac{37}{4}$ to both sides and continue as before.
Example 5: Solve $9x^2 + 10x + 1 = 0$ by completing the square.

Solution 1: Since the coefficient of $x^2$ is not 1, we have to be more careful when writing the left-hand side as $x^2 + kx$:

\[
x^2 + \frac{10}{9}x = -\frac{1}{9}
\]

\[
x^2 + \frac{10}{9}x + \frac{25}{81} = -\frac{1}{9} + \frac{25}{81}
\]

\[
x^2 + \frac{10}{9}x + \frac{25}{81} = \frac{16}{81}
\]

\[
(x + \frac{5}{9})^2 = \frac{16}{81}
\]

\[
x + \frac{5}{9} = \pm \frac{4}{9}
\]

\[
x = -\frac{5}{9} \pm \frac{4}{9}
\]

\[
x = -\frac{5}{9} + \frac{4}{9} = -\frac{1}{9} \text{ or } x = -\frac{5}{9} - \frac{4}{9} = -1
\]

Solution 2: You must also be extremely careful if you use the alternate method of completing the square here:

\[
(9x^2 + 10x) + 1 = 0
\]

\[
9\left(x^2 + \frac{10}{9}x\right) + 1 = 0
\]

\[
9\left(x^2 + \frac{10}{9}x + \left(\frac{5}{9}\right)^2\right) + 1 - 9\left(\frac{5}{9}\right)^2 = 0
\]

\[
9\left(x + \frac{5}{9}\right)^2 - \frac{16}{9} = 0
\]

Then proceed as above.

Question: Why $-9\left(\frac{5}{9}\right)^2$ in step 3?

Answer: We added the $\frac{5}{9}$ inside the parentheses, so the 9 distributes over it:

\[
9\left(x^2 + \frac{10}{9}x + \left(\frac{5}{9}\right)^2\right) = 9x^2 + 10x + 9\left(\frac{5}{9}\right)^2
\]

so we have actually added $9\left(\frac{5}{9}\right)^2$ to our expression. To balance this, we must subtract $9\left(\frac{5}{9}\right)^2$ from the same side or add it to the other side.
III. Solving with the Quadratic Formula

We can use the technique of completing the square on the general quadratic equation $ax^2 + bx + c = 0$ to derive a general formula for the solutions to a quadratic equation:

\[
ax^2 + bx + c = 0 \\
ax^2 + bx = -c \\
x^2 + \frac{b}{a}x = -\frac{c}{a} \\
x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = -\frac{c}{a} + \frac{b^2}{4a^2} \\
x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{-c(4a) + b^2}{a(4a)} \\
x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{b^2 - 4ac}{4a^2} \\
\left( x + \frac{b}{2a} \right)^2 = \frac{b^2 - 4ac}{4a^2} \\
x + \frac{b}{2a} = \pm \sqrt{\frac{b^2 - 4ac}{4a^2}} \\
x + \frac{b}{2a} = \pm \frac{\sqrt{b^2 - 4ac}}{2a} \\
x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} \\
x = -\frac{b \pm \sqrt{b^2 - 4ac}}{2a}
\]

The final step gives us the Quadratic Formula.

The solution to any quadratic equation of the form

\[
ax^2 + bx + c = 0
\]

is given by

\[
x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\]

This formula must be memorized.

Remember: $a$ is the coefficient of $x^2$, $b$ is the coefficient of $x$, and $c$ is the constant term in this formula.
Example 6: Solve \( x^2 + 7x + 3 = 0 \) using the Quadratic Formula

Solution: For the quadratic formula \( a = 1, b = 7, \) and \( c = 3. \)

\[
x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\]

Substitute in formula

\[
x = \frac{-7 \pm \sqrt{49 - 12}}{2}
\]

\[
x = \frac{-7 \pm \sqrt{37}}{2}
\]

Thus, we have

\[
x = \frac{-7 + \sqrt{37}}{2} \approx -0.45862 \text{ or } \frac{-7 - \sqrt{37}}{2} \approx -6.5414
\]

Example 7: Solve \( 5x^2 = x - 1 \) using any method.

Solution: First write the equation in standard form. The quadratic formula is generally the easiest method to use when the quadratic does not easily factor.

\[5x^2 = 4x - 1\]

\[5x^2 - 4x + 1 = 0\]

For the quadratic formula, \( a = 5, b = -4 \) (be careful with signs), and \( c = 1. \)

\[
x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(5)(1)}}{2(5)}
\]

\[
x = \frac{4 \pm \sqrt{-20}}{10}
\]

Simplify the complex radical (*See Note Below)

\[
x = \frac{4 \pm 2i \sqrt{5}}{10}
\]

Now factor and cancel the common 2

\[
x = \frac{2(2 \pm i \sqrt{5})}{10}
\]

\[
x = \frac{2 \pm i \sqrt{5}}{5}
\]

Note that the solutions above are complex numbers. Be sure to simplify any radicals in your final solutions!

*Note: In simplifying the complex radical, recall that

\[
i = \sqrt{-1}
\]

So

\[
\sqrt{-20} = \sqrt{4 \cdot 5 \cdot (-1)}
\]

\[
= \sqrt{4} \cdot \sqrt{5} \cdot \sqrt{-1}
\]

\[
= 2\sqrt{5}i
\]

\[
= 2i\sqrt{5}
\]