Solving Equations

Absolute Value Equations

The method for solving equations which contain the absolute value function comes from the definition of absolute value. Recall that

\[ |x| = \begin{cases} 
  x & \text{if } x \geq 0 \\
  -x & \text{if } x < 0 
\end{cases} \]

Therefore, for the absolute value of an expression to equal a number, either the expression or the negative of the expression must equal that number. To solve such equations we rewrite every absolute value equation as two separate equations.

**Example 1**: Solve \(|x| = 5\).

Solution: There are two cases to consider the first case is \(x \geq 0\) and the second is \(x < 0\). If \(x \geq 0\), then

\[ |x| = x = 5 \]

So \(x = 5\) is a solution, but we need to be careful here. Is 5 \(\geq 0\)? Remember we got the solution by assuming that \(x \geq 0\) so we need to check that our solution satisfies this condition.

For the other case, if \(x < 0\)

\[ |x| = -x = 5 \]

\[ x = -5. \]

Notice that in this case the solution \(x = -5\) satisfies the assumption that \(x < 0\). Thus, we have two solutions \(x = \pm 5\).

**Example 2**: Solve \(|y - 7| = 3\)

Solution: We again consider the cases where the argument of the absolute value function is greater or equal than zero, or less than zero. That is \(y - 7 \geq 0\) or \(y - 7 < 0\). We rewrite the equation into two equations: 

\[ y - 7 = 3 \quad \text{and} \quad -(y - 7) = 3 \]

(Remember that we must change all or none of the signs of the expression! \(|y - 7|\) is NOT the same as \(y + 7!!!\))

\[
\begin{align*}
\text{case 1: } & y - 7 \geq 0 & \text{case 2: } & y - 7 < 0 \\
& |y - 7| = y - 7 = 3 & & |y - 7| = -(y - 7) = 3 \\
& y = 10 & & -y + 7 = 3 \\
& & & -y = -4 \\
& & & y = 4
\end{align*}
\]

Notice: in both cases the solution minus 7 satisfies the appropriate assumption. That is, \(10 - 7 \geq 0\) and \(4 - 7 < 0\).

Let’s check our solutions:

|10 - 7| = 3 ? |4 - 7| = 3 ? \\
|3| = 3 ? |3| = 3 ? \\
3 = 3 \checkmark |3 = 3 \checkmark |

Both solutions are valid.
A geometrical interpretation of absolute value inequalities.

Recall that $|a - b|$ refers to the distance between the points $a$ and $b$ on the number line. Therefore, the equation $|x - 7| = 3$
can be thought of as, “the distance between $x$ and 7 is 3?”, or more appropriately, ”what number or numbers are
3 units from 7 on the number line?” See the illustration below:

Example 3: Solve the equation $|x - 1| = 2$.
Solution: As in the previous examples we break the problem into two cases.

Case 1: $x(x - 1) \geq 0$

$|x(x - 1)| = x(x - 1) = 2$
$x^2 - x - 2 = 0$
$(x + 1)(x - 2) = 0$
Thus, we have two solutions $x = -1$ and $x = 2$.

Case 2: $x(x - 1) < 0$

$|x(x - 1)| = -(x(x - 1)) = 2$
$x^2 - x + 2 = 0$
$x = \frac{1 \pm \sqrt{1 - 8}}{2}$
$x = \frac{1}{2} \pm \frac{\sqrt{7}}{2}i$

In this case there are no real solutions. Thus, the final answer is $x = -1$ and $x = 2$. 