Solving Inequalities

Linear Inequalities
To solve a linear inequality, isolate the variable in the same fashion as solving a linear equation. However, there is one major difference: Whenever you multiply or divide an inequality by a negative number, you must switch the inequality symbol (from < to > or vice versa).

Why? Consider the following example:

\[ 5 < 7 \]

If we multiply both sides of the inequality by \(-1\), the result is

\[ -5 > -7 \]

This is due to the fact that, on the number line, the negative numbers are written “backwards”, namely, the smaller the absolute value, the larger the number.

Example 1: Solve for \(x\):

\[ 7 - 2x < 11 \]

Solution: Solve for \(x\) the same way you would solve the equation \(7 - 2x = 11\):

\[
\begin{align*}
7 - 2x &< 11 \\
-2x &< 4 \\
\frac{-2x}{-2} &> \frac{4}{-2} \\
x &> -2
\end{align*}
\]

The solution can be written algebraically (\(x > -2\)), using interval notation \((-2, \infty)\) or a number line which is shown below.

Example 2: Solve for \(y\):

\[ \frac{y - 1}{3} - \frac{3y}{5} > -1 \]

Solution: It is best to multiply by the LCD (least common denominator) first, which is 15:

\[
\begin{align*}
15 \left( \frac{y - 1}{3} \right) - 15 \left( \frac{3y}{5} \right) &\geq 15(-1) \\
5(y - 1) - 3(3y) &\geq -15 \\
5y - 5 - 9y &\geq -15 \\
-4y - 5 &\geq -15 \text{ add 5} \\
-4y &\geq -10 \text{ divide by } -4 \\
y &\leq \frac{-10}{-4} = \frac{5}{2} \\
(-\infty, \frac{5}{2})
\end{align*}
\]

Notice how we indicate that \(5/2\) is included in the solution set by using a solid dot. If we want to indicate that \(5/2\) is not included an unfilled circle will be used.
Sometimes two inequalities are combined in the same statement, such as \( a < x < b \). As stated earlier, this means the number \( x \) is between \( a \) and \( b \). More formally, we say that
\[
a < x < b
\]
Note: in order for this to be true, \( a \) must be less than \( b \) to start with. A statement such as \( 4 < x < -2 \) cannot be true; there are no numbers greater than 4 and simultaneously less than \(-2\).
Linear inequalities of this type may be solved by simply splitting them into two inequalities, solving each separately, and finding all values common to both solutions.

**Question:** Is there anything wrong with the statement \( 5 < x < 5 \)?
**Answer:** Yes, there is no number which is both less than 5 and greater than 5.

**Example 3:** Solve for \( x \):
\[-10 < 3x + 5 \leq 17\]
**Solution 1:** The combined inequality can be rewritten as
\[
\begin{align*}
3x + 5 &> -10 \\
3x + 5 &\leq 17
\end{align*}
\]
subtract 5 from both sides
\[
\begin{align*}
3x &> -15 \\
x &> -5
\end{align*}
\]
subtract \( x \) from both sides
\[
\begin{align*}
x &\leq 4
\end{align*}
\]
divide both sides by 3
The solution is all real numbers greater than \(-5\) and less than or equal to \(4\). This can be written as \(-5 < x \leq 4\), or \((-5, 4]\), or we can use a number line as shown below.

As usual the red line is used to indicate the solution set, and filled and unfilled circles indicated whether a point is in the solution set or not.

**Solution 2:** Since the variable appears only in the middle, we can isolate it as we do for a single inequality. Remember to do the same operations to all 3 parts and switch the inequality when necessary!
\[
\begin{align*}
-10 &< 3x + 5 \leq 17 \\
-15 &< 3x \leq 12 \\
-5 &< x \leq 4
\end{align*}
\]
As found earlier.

**Example 4:** Solve for \( x \):
\[x + 4 < 2x - 7 < 3 - 3x\]
**Solution:** The combined inequality can be rewritten as
\[
\begin{align*}
2x - 7 &> x + 4 \\
2x &> x + 11 \\
x &> 11
\end{align*}
\]
subtract \( x \) from both sides
\[
\begin{align*}
2x - 7 &< 3 - 3x \\
2x &< 10 - 3x \\
5x &< 10 \\
x &< 2
\end{align*}
\]
add 7 to both sides
add 3x to both sides
divide both sides by 5

Note that the solution is all real numbers greater than \(11\) and less than \(2\). Since no real numbers satisfy both statements, there is no solution to the inequality. Note also that these two inequalities cannot be solved together since there is no way to isolate \( x \) in the center.