Systems of Two Equations

Geometry of Solutions

In an earlier chapter we learned how to solve a single equation in one unknown. The general form of such an equation has the form

\[ ax = b, \]

where the constants \( a \neq 0 \) and \( b \) are assumed known, and we are looking for a value of \( x \) which satisfies the equation. Since \( a \neq 0 \), the equation is easy to solve. Multiply by \( a^{-1} \). Thus, we have the solution \( x = \frac{b}{a} \).

The situation is not quite so simple when we have more than one equation and unknown. First we give the standard form for a system of two equations in two unknowns.

\[
\begin{align*}
ax + by &= c \\
dx + ey &= f,
\end{align*}
\]

where the constants \( a, b, c, d, e, \) and \( f \) are assumed known.

By a solution of this system we mean a pair of numbers \( x_0 \) and \( y_0 \) which satisfy the system of equations. That is, when the substitutions \( x = x_0 \) and \( y = y_0 \) are made in the system both equations become identities.

This pair of numbers is commonly written as \( (x_0, y_0) \) and interpreted as a point in the Euclidean plane, \( R^2 \).

By the solution set of a system we mean the totality of all possible solutions to the system.

Example 1: Which of the pairs of numbers \((1, 2), (11, 3), \) or \((-19, -9)\) are solutions to the given system?

\[
\begin{align*}
2x - 5y &= 7 \\
-x + 2y &= 1.
\end{align*}
\]

Solution: To check that a pair of numbers is a solution we substitute the values in for \( x \) and \( y \). In the first equation if we substitute \( x = 1 \) and \( y = 2 \) we have

\[
2x - 5y = 7 \rightarrow 2(1) - 5(2) = 7 \rightarrow -8 = 7
\]

Since \(-8 \neq 7\) the first pair does not satisfy the first equation let alone both equations.

We try the second pair \((11, 3)\) next:

\[
2x - 5y = 7 \rightarrow 2(11) - 5(3) = 7 \rightarrow 7 = 7
\]

Okay, the second pair satisfies the first equation, but that is not enough to guarantee that we have a solution to the system. The second equation still has to be checked, which we do below.

\[
-x + 2y = 1 \rightarrow -(11) + 2(3) = 1 \rightarrow -5 = 1
\]

Since \(-5 \neq 1\), this pair does not solve the system. Now let’s check the last pair \((-19, -9)\) to see if it is a solution

\[
2x - 5y = 7 \rightarrow 2(-19) - 5(-9) = 7 \rightarrow 7 = 7
\]

\[
2x - 5y = 7 \rightarrow 2(-19) - 5(-9) = 7 \rightarrow 7 = 7
\]

Since the pair \((-19, -9)\) satisfies both equations, this pair is a solution to the system.
Question: Does the pair of numbers 1, -2 satisfy the system?
\[
2x - y = 3 \\
-3x + 6y = 3
\]
Answer: No, the first equation is satisfied, but the second is not.

Note: There is no reason why a system must consist of two equations. In the following pages we will have examples of systems which consist of a single equation with more than one unknown, systems which consist of two equations with three unknowns, and finally the general system which consists of \( n \) equations in \( m \) unknowns. In this later case there need be no a-priori relationship between the sizes of \( m \) and \( n \).

Example 2: Show that the pair of numbers \((-3, 2)\) satisfies the system
\[
2x + 7y = 8 \\
-x + y = 5
\]
Solution: To verify that \((-3, 2)\) is a solution we just substitute \(-3\) for \( x \) and \( 2 \) for \( y \) in each equation and then check that we have an identity.
\[
2x + 7y = 8 \rightarrow 2(-3) + 7(2) = 8 \rightarrow 8 = 8 \\
-x + y = 5 \rightarrow -(3) + (2) = 5 \rightarrow 5 = 5
\]
Since both equations are true the pair \((-3, 2)\) is a solution of the system.

Example 3: Consider the system
\[
-x + 3y = 2 \\
x + y = 1
\]
Is there a number \( y_0 \) so that the pair \((1, y_0)\) is a solution to the system?
Solution: If the pair \((1, y_0)\) is a solution to the system then \( y_0 \) must satisfy the first equation and the second equation with 1 substituted for \( x \).
\[
-1 + 3y_0 = 2 \\
1 + y_0 = 1
\]
Solving the first equation for \( y_0 \) and then the second equation for \( y_0 \) we have
\[
y_0 = \frac{2 + 1}{3} = 1 \quad \text{from the first equation} \\
y_0 = 1 - 1 = 0 \quad \text{from the second equation}
\]
Thus, there is no value of \( y_0 \) which can satisfy both equations, and we conclude there is no pair of numbers of the form \((1, y_0)\) which satisfies this system.

Example 4: Is there a number \( x_0 \) such that the pair \((x_0, -2)\) satisfies the system
\[
3x + y = 4 \\
5x - 3y = 16
\]
Solution: As in the previous example we substitute \(-2\) for \( y \) in both equations and see if there is a single value of \( x \) which will satisfy both equations simultaneously.
\[
3x - 2 = 4 \Rightarrow 3x = 6 \Rightarrow x = 2 \\
5x - 3(-2) = 16 \Rightarrow 5x = 16 - 6 = 10 \Rightarrow x = 2
\]
The value \( x = 2 \) works. Thus, the pair \((2, -2)\) satisfies the system.
In this page we begin our study of the geometry of solutions to systems of equations. First we examine the solution set for the single equation in two unknowns
\[ 2x - 3y = 1. \]

As we have seen, the locus of points in the plane whose coordinates satisfy an equation of this sort is a straight line. The graph of the solution set of the equation \( 2x - 3y = 1 \) is shown below.

What can we say about the geometry of solutions to the system
\[ \begin{align*}
2x - 3y &= 1 \\
x + y &= 2
\end{align*} \]

If we have a solution to this system then we know that it must lie on the straight line shown above, since the solution to the system is a solution to the first equation. However, it is also a solution to the second equation. Thus, the solution must also lie on the straight line \( x + y = 2 \). Hence, any solution to the system must lie on the intersection of both lines. The two lines are shown below. Their point of intersection is \((7/5, 3/5)\).

Question: Verify that \((7/5, 3/5)\) is a solution to this system.
Solution: \( 2x - 3y = 1 \Rightarrow 2(7/5) - 3(3/5) = 1 \Rightarrow 1 = 1 \). \( x + y = 2 \Rightarrow (7/5) + 3/5 = 2 \Rightarrow 2 = 2 \)

There are three possibilities for the configuration of two lines in a plane. The lines intersect in a unique point, as they do above; the lines are parallel and not equal to each other, which means they have no points of intersection; the third possibility is that the two lines are the same line, in which case there are an infinite number of points of intersection. This leads us to believe that for a linear system of equations the solution set can have one of the following three characterizations
1. The solution set consists of a single solution.
2. The solution set is empty, that is there are no solutions.
3. There are an infinite number of solutions.

Question: How many solutions does the following system have, one, none, or an infinite number?
\[ \begin{align*}
5x - 3y &= 2 \\
10x - 6y &= 4
\end{align*} \]

Answer: Since the second equation is a multiple of the first equation, every solution of the first equation will also be a solution of the second equation. Moreover, the first equation has an infinite number of solutions. Thus, the system has an infinite number of solutions.
The plots below exhibit the three possibilities for the number of solutions to a linear system of equations.

- **A unique solution**
  
  \[
  \begin{align*}
  x - y &= 1 \\
  x + y &= 1 \\
  \end{align*}
  \]
  
  A unique solution \( x = 1, y = 0 \).

- **No solutions, the lines are parallel**
  
  \[
  \begin{align*}
  2x + y &= 1 \\
  2x + y &= 3 \\
  \end{align*}
  \]
  
  No solutions, the lines are parallel.

- **An infinite number of solutions**
  
  \[
  \begin{align*}
  3x + 2y &= 6 \\
  6x + 4y &= 12 \\
  \end{align*}
  \]
  
  An infinite number of solutions.

If we want to find the solution set to a system of equations in two unknowns, we can do so by graphing the straight line which corresponds to a particular equation in our system. However, this is not quickly done, nor is it accurate. In the next sections we will describe algebraic ways of solving a system of equations.

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**Example 5:** Graph the solution set of the system, and from your plot describe the solution set to this system.

\[
\begin{align*}
  x + y &= 2 \\
  2x - 3y &= 1 \\
  x - y &= 4 \\
  \end{align*}
\]

Solution:

We note that each pair of lines has a single point of intersection, but there is no point which lies on all three lines. Hence, this system has no solution. Its solution set is empty.