Systems of Two Equations

Algebraic Methods of Solution

There are essentially two algebraic methods to solve a system of equations. The first, which is called the substitution method, is easy to use. However, its use is usually relegated to very simple systems. The second method, Gaussian elimination, is almost always a better method than substitution.

The Method of Substitution

Pick one of the equations, solve this equation for one of the unknowns in terms of the other unknown(s), substitute the expression for the solved unknown in the remaining equations. Repeat this process until one of the equations has been reduced to an equation in only one unknown. Solve for this unknown, and then use this value to determine the values of the other unknowns.

The next few examples demonstrate how to solve a system of equations using the method of substitution.

Example 1: Solve the linear system of equations

\[
\begin{align*}
x - y &= 4 \\
2x + 5y &= 8.
\end{align*}
\]

Solution: Solve the first equation for \( y \) in terms of \( x \). (We could just as easily solve for \( x \) in terms of \( y \).)

\[
x - y = 4 \rightarrow y = x - 4.
\]

Substitute \( x - 4 \) for \( y \) in the second equation, and then solve for \( x \).

\[
\begin{align*}
2x + 5y &= 8 \\
2x + 5(x - 4) &= 8 \\
2x + 5x - 20 &= 8 \\
7x - 20 &= 8 \\
7x &= 28 \\
x &= 4
\end{align*}
\]

Take this value of \( x \) and calculate \( y \).

\[
\begin{align*}
y &= x - 4 \\
y &= (4) - 4 \\
y &= 0
\end{align*}
\]

Thus, the pair \( (4, 0) \) is a solution to this system. You can easily verify that the pair \( (4, 0) \) really is a solution to the system.

Example 2: Use the method of substitution to solve the system

\[
\begin{align*}
x - 3y &= 6 \\
2x + 5y &= 7
\end{align*}
\]

Solution: Solve the first equation for \( x \) (That seems easier than solving for \( y \)), and then substitute this expression for \( x \) into the second equation.
\[ x - 3y = 6 \rightarrow x = 6 + 3y \]
\[ 2x + 5y = 7 \rightarrow \]
\[ 2(6 + 3y) + 5y = 7 \]
\[ 12 + 6y + 5y = 7 \]
\[ 12 + 11y = 7 \]
\[ 11y = 12 - 7 \]
\[ y = \frac{-5}{11} \]
\[ x = 6 + 3y = 6 + 3\left( \frac{-5}{11} \right) = 6 - \frac{15}{11} = \frac{51}{11} \]

The solution is \( \left( \frac{51}{11}, \frac{-5}{11} \right) \).

**Question:** Consider the system

\[ 6x + 17y = 9 \]
\[ 2x + y = 6 \]

Which equation would you solve for one of the variables, and which variable would you solve for?

**Answer:** Solve the second equation for \( y \), and substitute this expression for \( y \) into the first equation.

**Example 3:** Solve the following system using the method of substitution. Then plot the equations and the solution.

\[ 6x + 17y = 9 \]
\[ 2x + y = 6 \]

**Solution:** First solve the second equation for \( y \) in terms of \( x \), to get \( y = 6 - 2x \). Substitute this expression into the first equation and solve for \( y \).

\[ 6x + 17y = 9 \]
\[ 6x + 17(6 - 2x) = 9 \]
\[ 6x + 102 - 34x = 9 \]
\[ -28x = 9 - 102 = -93 \]
\[ x = \frac{-93}{-28} = \frac{93}{28} \approx 3.32143 \]

Thus, \( y = 6 - 2x = 6 - 2\left( \frac{93}{28} \right) = -\frac{9}{14} \approx -0.642857 \). Thus, the solution is \( \left( \frac{93}{28}, -\frac{9}{14} \right) \approx (3.321, -0.643) \).

The plot of the solution set is given below.
Example 4: Solve the following system of equation using the method of substitution.

\[
\begin{align*}
2x - 6y &= 5 \\
-x + 3y &= 4
\end{align*}
\]

Solution: Solve the second equation for \( x \), and then substitute into the first equation.

\[
\begin{align*}
-x + 3y &= 4 \\
2x - 6y &= 5 \\
6y - 8 - 6y &= 5 \\
-8 &= 5
\end{align*}
\]

Something bad has happened. After substituting \( 3y - 4 \) for \( x \) in the first equation the terms involving \( y \) cancel, and we are left with a contradiction \(-8 = 5\). This means that there is no solution for this system.

If we go back and examine the system, we soon see that the left side of the second equation is a multiple of the first equation.

\[
2x - 6y = 5 \text{ multiply by } \frac{-1}{2} \rightarrow -x + 3y = \frac{-5}{2}
\]

Thus, any solution to our system must satisfy the two equations

\[
\begin{align*}
-x + 3y &= \frac{-5}{2} \\
-x + 3y &= 4
\end{align*}
\]

Clearly this cannot happen. Geometrically these equations represent two parallel lines which are plotted below.

Example 5: Solve the following system using substitution.

\[
\begin{align*}
3x - 15y &= 18 \\
-6x + 30y &= -36
\end{align*}
\]

Solution: Solving the first equation for \( x \) we have

\[
\begin{align*}
3x - 15y &= 18 \\
3x &= 18 + 15y \\
x &= 6 + 5y
\end{align*}
\]

Substitute this expression for \( x \) into the second equation

\[
\begin{align*}
-6x + 30y &= -36 \text{ substitute for } x \\
-6(6 + 5y) + 30y &= -36 \text{ simplify the left hand side} \\
-36 &= -36
\end{align*}
\]

This last line implies that no matter what value \( y \) has as long as we set \( x = 6 + 5y \) we will have a solution. Thus, the solution set for this system consists of all pairs of numbers \((6 + 5y, y)\) for any value of \( y \).
We have seen how to solve a system using the process of substitution. However, this method is highly ineffective when the number of equations starts getting large. For the method of substitution large typically means more than 2. The second method (elimination), which we consider next, is somewhat more complicated, but it is a lot more efficient, and better suited to handle larger numbers of equations and unknowns.

The idea behind the method of elimination is to algebraically manipulate the equations in order to reduce the number of variables in some of the equations. There are three algebraic manipulations allowed, and they are:

**Elementary Row (Equation) Operations:**
1. Interchange the order in which the equations are written.
2. Multiply an equation by any non-zero number.
3. Add a multiple of one equation to another equation.

Note: the reason for using the word row will become apparent when we introduce matrices and Gaussian elimination.

We examine what each of these operations does to a system of equations in the following example.

**Example 6:** Consider the following system of 3 equations in 4 unknowns

\[
\begin{align*}
2x - 3y + z - w &= 1 \\
-x - y + 4z - 3w &= 0 \\
x + 6y - w &= 5
\end{align*}
\]

Interchange Eq1 and Eq3:

\[
\begin{align*}
-x - y + 4z - 3w &= 0 \\
x + 6y - w &= 5 \\
2x - 3y + z - w &= 1
\end{align*}
\]

Note, if we interchange equations 1 and 3 once more we will return to our original system.

Multiply Eq1 by \( \frac{1}{2} \):

\[
\begin{align*}
-x - y + 4z - 3w &= 0 \\
x + 6y - w &= 5 \\
2x - 3y + z - w &= 1
\end{align*}
\]

Note, if we multiply the new first equation by 2, we will return to the original system.

Add \(-2\)Eq3 to Eq1:

\[
\begin{align*}
-x - y + 4z - 3w &= 0 \\
x + 6y - w &= 5
\end{align*}
\]

Notice how this last operation eliminated the \( x \) variable from the first equation. This is the essential use of the third elementary row (equation) operation.

Note too, if, in the new system, we now add 2 times equation 3 to equation 1 we will return to the original system.
Example 7: Solve the system below by performing a sequence of elementary row operations.

\[
\begin{align*}
2x + y &= 8 \\
x - y &= 6
\end{align*}
\]

Interchange equations 2 and 1.

\[
\begin{align*}
x - y &= 6 \\
2x + y &= 8
\end{align*}
\]

Add \(-2\) times equation 1 to equation 2. Be sure you understand this one, and notice its effect on the system; \(x\) is eliminated from the second equation. Simplify

\[
\begin{align*}
x - y &= 6 \\
3y &= -4
\end{align*}
\]

Multiply equation 2 by \(\frac{1}{3}\). We are solving for \(y\).

\[
\begin{align*}
x - y &= 6 \\
y &= -\frac{4}{3}
\end{align*}
\]

Add 1 times equation 2 to equation 1. To eliminate \(y\) from the first equation.

\[
\begin{align*}
x - y + y &= 6 + \left( -\frac{4}{3} \right) \\
y &= -\frac{4}{3}
\end{align*}
\]

Simplify

\[
\begin{align*}
x &= \frac{14}{3} \\
y &= -\frac{4}{3}
\end{align*}
\]

Thus, the solution to this system is the pair \(\left( \frac{14}{3}, -\frac{4}{3} \right)\). Something peculiar has just occurred. Why should \(\left( \frac{14}{3}, -\frac{4}{3} \right)\) be a solution to the original system we started out with? The answer lies in the fact that all three of the elementary row operations are reversible. That is, if we have a system, call it system \(A\), and we perform an elementary row operation to transform system \(A\) into a second system, then there is a second elementary row operation which will transform the second system back into system \(A\). This observation is justification for the following theorem.

**Theorem:** Suppose that we have two systems of linear equations, system \(A\) and system \(B\). Suppose system \(B\) is obtained from system \(A\) by a sequence of elementary row (equation) operations, then the solution set of the first system is the same as the solution set of the second system. That is, something solves system \(A\) if and only if it also solves system \(B\).

**Question:** What system do we get if we take \(-2\) times equation 1 and add it to equation 3 in the following system?

\[
\begin{align*}
2x - 5y + z &= 0 \\
-x - y - 3z &= 5 \\
4x + y + 13z &= 1
\end{align*}
\]

**Answer:**

\[
\begin{align*}
2x - 5y + z &= 0 \\
-x - y - 3z &= 5 \\
11y + 11z &= 1
\end{align*}
\]
Example 8: Use the method of elimination to solve the following system of equations

\[ \begin{align*}
2x + y &= 7 \\
-4x - y &= 6
\end{align*} \]

Solution: It may seem that the thing to do is to add the two equations together to eliminate the \( y \) variable, but we do not do that. Instead we eliminate the \( x \) variable from the second equation. The reason being to demonstrate the method of elimination rather than something that works for a special equation.

\[ \begin{align*}
\text{Add} \ 2 \ \text{Eq} \ 1 \ \text{to Eq} \ 2: \ 2x + y &= 7 \quad \Rightarrow \quad 2x + y = 7 \\
&\quad -4x - y = 6 \quad y = 20
\end{align*} \]

From this last equation we see that \( y = 20 \). Now we can either substitute \( y = 20 \) into the first equation and solve for \( x \), or we can use the elementary row operations to determine \( x \). Since we’re trying to demonstrate how to use the elementary row operations, we’ll do the later.

\[ \begin{align*}
\text{Add} \ -\text{Eq} \ 2 \ \text{to Eq} \ 1: \ 2x + y &= 7 \quad \Rightarrow \quad 2x = -23 \\
&\quad y = 20 \quad y = 20
\end{align*} \]

\[ \begin{align*}
\text{Multiply} \ \text{Eq} \ 1 \ \text{by} \ \frac{1}{2}: \ 2x &= -23 \quad \Rightarrow \quad x = \frac{-23}{2} \\
&\quad y = 20 \quad y = 20
\end{align*} \]

Thus, we have found the solution \( \left( \frac{-23}{2}, 20 \right) \).

Example 9: Solve the following system of equations using the method of elimination.

\[ \begin{align*}
3x - 5y &= 16 \\
x + y &= 2
\end{align*} \]

Solution: In the following we will start out with the given system of equations, perform a row operation and ask you, the reader, to identify which elementary row (equation) operation was used.

\[ \begin{align*}
3x - 5y &= 16 \quad \Rightarrow \quad x + y = 2 \\
x + y &= 2 \quad 3x - 5y = 16
\end{align*} \]

Question: Which row operation was used? Answer: Equations 1 and 2 were interchanged.

\[ \begin{align*}
x + y &= 2 \quad \Rightarrow \quad x + y = 2 \\
3x - 5y &= 16 \quad -8y = 10
\end{align*} \]

Question: Which row operation was used? Answer: \(-3\) times the first equation was added to the second equation.

\[ \begin{align*}
x + y &= 2 \quad \Rightarrow \quad x + y = 2 \\
-8y &= 10 \quad y = \frac{-5}{4}
\end{align*} \]

Question: Which row operation was used? Answer: \(-\frac{1}{8}\) times equation 2

\[ \begin{align*}
x + y &= 2 \quad \Rightarrow \quad x = \frac{13}{4} \\
y &= \frac{-5}{4} \quad y = \frac{-5}{4}
\end{align*} \]

Question: Which row operation was used? Answer: The negative of the second equation was added to the first.

The solution to this last system of equations is \( \left( \frac{13}{4}, \frac{-5}{4} \right) \). Hence by the theorem which states that the last system and the original system have the same set of solutions, we know that \( \left( \frac{13}{4}, \frac{-5}{4} \right) \) is the solution to the original system of equations. Since the last system clearly has only one solution, we know that the original system has only one solution.
Example 10: Solve the following system by the process of elimination.

\[
5x + 2y = 8 \\
3x + y = 5
\]

Solution: Work the example yourself before comparing your work with the author’s.

Question: What is the first row operation you would use?

Author’s Answer: The authors decided to multiply equation 1 by \( \frac{1}{5} \). Their reason for doing so was to have the coefficient of \( x \) in the first equation equal to 1. If you picked something else, that is certainly okay. After performing the author’s row operation the original system

\[
5x + 2y = 8 \\
3x + y = 5
\]

is transformed into the system

\[
x + \frac{2}{5}y = \frac{8}{5} \\
3x + y = 5
\]

Question: Given this system what row operation would you perform next?

Author’s Answer: The author’s decided to add \(-3\) times equation 1 to the second equation. Thus eliminating the \( x \) variable from the second equation. The last system

\[
x + \frac{2}{5}y = \frac{8}{5} \\
3x + y = 5
\]

then becomes

\[
x + \frac{2}{5}y = \frac{8}{5} \\
-\frac{1}{5}y = \frac{1}{5}
\]

Question: Given this system what row operation would you perform next?

Author’s Answer: We decided to multiply the second equation by \(-5\). This transforms the system

\[
x + \frac{2}{5}y = \frac{8}{5} \\
-\frac{1}{5}y = \frac{1}{5}
\]

into the system

\[
x + \frac{2}{5}y = \frac{8}{5} \\
y = -1
\]

Question: Given this system what row operation would you perform next?

Author’s Answer: The last operation is \(-2\) times the second equation added to the first equation, eliminating the \( y \) variable from the first equation. So the system

\[
x + \frac{2}{5}y = \frac{8}{5} \\
y = -1
\]

is transformed into the system

\[
x = 2 \\
y = -1
\]

Therefore the solution to the original system is \( x = 2 \) and \( y = -1 \).
In the next few examples we will see how systems of equations can arise quite naturally. Pay particular attention to how the mathematical equations arise from the words which describe the problem.

Example 11: A plane flies a round trip between two cities. The flight from the first city is into a strong headwind and takes 1 hour and 30 minutes. The return flight is with the wind and take 55 minutes. If the cities are 100 miles apart what is the aircraft’s speed, and what is the wind’s speed. Assume that both the aircraft’s and wind’s speeds are constant.

Solution: Let $p$ denote the aircraft’s speed in miles per hour and let $w$ denote the wind’s speed also in miles per hour. The basic fact which we need here is that distance equals rate times time. In that part of the trip against the wind, the speed of the plane is $\frac{p-w}{u2212w}$ and the plane’s speed with the wind is $\frac{p+w}{u002b+w}$. This leads to the following system of equations.

$$\frac{3}{2}(p-w) = 100$$
$$\frac{55}{60}(p+w) = 100$$

Note that we’ve converted the flight times from minutes to hours, so that the speeds we solve for will be in miles per hour and not miles per minute.

The above system leads to the following system

$$p-w = \frac{200}{3}$$
$$p+w = \frac{6000}{55}$$

Adding the two equations together we get

$$2p = \frac{200}{3} + \frac{6000}{55} = \frac{5800}{33}$$
$$p = \frac{5800}{66} = \frac{2900}{33} \approx 87.89 \text{ miles per hour}$$

To determine the wind’s speed we have from the second equation

$$w = \frac{6000}{55} - p$$
$$= \frac{6000}{55} - \frac{2900}{33}$$
$$= \frac{700}{33}$$
$$\approx 21.21 \text{ miles per hour}$$

Example 12: A salad dressing manufacturer wants to make a new version of a honey mustard dressing by combining two other honey mustard dressings. Dressing number 1 contains 5% honey and dressing number 2 contains 4% honey. How many quarts of each of these dressings must the manufacture combine in order to produce 1000 quarts of a 4.75% honey dressing?

Solution: Let $d_1$ represent the number of quarts used of the first dressing and $d_2$ the number of quarts of the second dressing. Thus,

$$d_1 + d_2 = 1000$$
$$0.05d_1 + 0.04d_2 = 0.0475(1000).$$

Solving the first equation for $d_2$ and substituting this expression into the second equation gives

$$0.05d_1 + 0.04(1000 - d_1) = 0.0475(1000)$$
$$5d_1 + 4(1000 - d_1) = 4750$$
$$d_1 + 4000 = 4750$$
$$d_1 = 750.$$