Matrix Notation For a System

Quite often the size of a system of equations which is used to model some physical problem is very large. Hundreds even thousands of equations and unknowns are not uncommon. In dealing with such systems it is helpful to have a common notational scheme. This notational scheme is described below. We start with a simple example, a system of two equations in two unknowns.

\[ \begin{align*} ax + by &= c \\ dx + ey &= f. \end{align*} \]

Looking at this system it’s clear that we will soon run out of letters with which to label the coefficients of the unknowns and letters with which to label the unknowns. The following question demonstrates another problem with the notational scheme used above.

Question: With out looking at the system, what is the role that \( e \) plays?

Answer: The letter \( e \) is the coefficient of the unknown \( y \) in the second equation. However, if you’re not looking at the system, that is not at all clear.

One cure for both of these problems is the following scheme:

1. All unknowns are denoted by a common letter, which is subscripted. For us this letter will usually be \( x \). The first unknown is \( x_1 \), the second \( x_2 \), and the 100th unknown is labeled \( x_{100} \).
2. All coefficients of unknowns will be denoted by a common doubly subscripted letter; usually \( a \). The subscripts denote the equation the coefficient is in as well as the unknown it multiplies. Thus, the coefficient of \( x_2 \) in the first equation is \( a_{12} \). The first subscript is the equation number, and the second subscript indicates which unknown is multiplied by this coefficient.
3. The constant to the right of the equals sign is also designated by a single subscripted letter. For us \( b_i \). The subscript indicating which equation.

We now rewrite the general linear system consisting of two equations in two unknowns.

\[ \begin{align*} a_{11}x_1 + a_{12}x_2 &= b_1 \\ a_{21}x_1 + a_{22}x_2 &= b_2. \end{align*} \]

**Example 1:** For the linear system below determine the \( a_{ij} \)’s and the \( b_i \)’s.

\[ \begin{align*} -2x_1 + 5x_2 &= 6 \\ 7x_1 + x_2 &= -9. \end{align*} \]

Solution:

<table>
<thead>
<tr>
<th>Equation</th>
<th>First Unknown</th>
<th>Second Unknown</th>
<th>Right Hand Side</th>
</tr>
</thead>
<tbody>
<tr>
<td>First Equation</td>
<td>( a_{11} = -2 )</td>
<td>( a_{12} = 5 )</td>
<td>( b_1 = 6 )</td>
</tr>
<tr>
<td>Second Equation</td>
<td>( a_{21} = 7 )</td>
<td>( a_{22} = 1 )</td>
<td>( b_2 = -9 )</td>
</tr>
</tbody>
</table>

The following graphic may help you keep the subscripts straight.

![Subscript graphic](image)

Question: In what equation will the coefficient \( a_{32} \) appear, and what variable will it multiply?

Answer: \( a_{32} \) is the coefficient of the second variable \( x_2 \) in the third equation.
Example 2: For the following system of two equations and three unknowns determine the values of the $a_{ij}$, $b_i$, and $x_i$.

\[
\begin{align*}
2x - 5z &= 7 \\
8x + 2y - 7z &= 21
\end{align*}
\]

Solution: The unknowns are labeled by $x_1 = x$, $x_2 = y$, and $x_3 = z$. The coefficients are labeled as follows:

<table>
<thead>
<tr>
<th>First Unknown</th>
<th>Second Unknown</th>
<th>Third Unknown</th>
<th>Right Hand Side</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_{11} = 2$</td>
<td>$a_{12} = 0$</td>
<td>$a_{13} = -5$</td>
<td>$b_1 = 7$</td>
</tr>
<tr>
<td>$a_{21} = 8$</td>
<td>$a_{22} = 2$</td>
<td>$a_{23} = -7$</td>
<td>$b_2 = 21$</td>
</tr>
</tbody>
</table>

Example 3: The following is a list of coefficients for a linear system of equations.

\[
\begin{align*}
a_{11} &= 5, \quad a_{12} = -6, \quad a_{22} = 3, \quad a_{21} = 11, \quad a_{13} = -4, \quad a_{23} = 71 \\
b_1 &= 87, \quad b_2 = 19
\end{align*}
\]

How many equations and unknowns does this system have? Write out the system of equations.

Solution: When we look at the double subscripts of the coefficients $a_{ij}$, we see the first subscript varies from 1 to 2. Hence there are two equations. The second subscript varies from 1 to 3. Thus, there are 3 unknowns. To further confirm that there are two equations we notice that the subscripts on the $b$ terms are 1 or 2. The system of equations is written below.

\[
\begin{align*}
5x_1 - 6x_2 - 4x_3 &= 87 \\
11x_1 + 3x_2 + 71x_3 &= 19
\end{align*}
\]

Example 4: Determine what $a_{ij}$ and $b_i$ equal for the following system of equations.

\[
\begin{align*}
-3x_1 + 12x_2 + 9x_4 &= -2 \\
x_1 - 5x_3 &= 0 \\
x_2 + 53x_3 - 8x_4 &= 18
\end{align*}
\]

Solution: Before starting to assign labels we note that there are 3 equations with 4 unknowns.

\[
\begin{align*}
a_{11} &= -3 & a_{12} &= 12 & a_{13} &= 0 & a_{14} &= 9 & b_1 &= -2 \\
a_{21} &= 1 & a_{22} &= 0 & a_{23} &= -5 & a_{24} &= 0 & b_2 &= 0 \\
a_{31} &= 0 & a_{32} &= 1 & a_{33} &= 53 & a_{34} &= -8 & b_3 &= 18
\end{align*}
\]

Example 5: \[
\begin{bmatrix}
1 & 0 \\
-8 & 6
\end{bmatrix}
\] is a $2 \times 2$ matrix. Two rows, two columns.

Example 6: \[
\begin{bmatrix}
-1 & 2 & -3 \\
-4 & 5 & 18
\end{bmatrix}
\] is a $2 \times 3$ matrix. Two rows, three columns.
Example 7: \[
\begin{bmatrix}
15 & 13 \\
7 & 9 \\
21 & 52
\end{bmatrix}
\]
is a 3 \times 2 matrix. Three rows, two columns.

If \(A\) is an \(m \times n\) matrix, then \(A\) has \(mn\) entries. Thus, a 2 \times 2 matrix has 4 entries, a 2 \times 3 and a 3 \times 2 matrix each have 6 entries. A 1 \times 5 matrix has one row with 5 entries and a 7 \times 1 matrix has one column with 7 entries. A matrix is commonly denoted by \(A = [a_{ij}]\), where the \(a_{ij}\) denote the individual entries of the matrix \(A\). As with systems of equations, the first subscript designates the row and the second subscript the column the entry appears in. The following represents an arbitrary \(m \times n\) matrix.

\[
A = \begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn}
\end{bmatrix}
\]

Example 8: In the matrix \[
\begin{bmatrix}
1 & 2 & 3 \\
4 & 5 & 6
\end{bmatrix}
\]
identify the entries using the \(a_{ij}\) notation.

Solution
\[
a_{11} = 1, \quad a_{12} = 2, \quad a_{13} = 3 \\
a_{21} = 4, \quad a_{22} = 5, \quad a_{23} = 6
\]

Example 9: In the following 4 \times 3 matrix, determine \(a_{23}, a_{32},\) and \(a_{13}\).

\[
\begin{bmatrix}
-2 & 0 & 9 \\
1 & 15 & -4 \\
-5 & 6 & 13 \\
-11 & 10 & 21
\end{bmatrix}
\]

Solution: \(a_{23} = -4, a_{32} = 6,\) and \(a_{13} = 9\).

Example 10: The entries for a 2 \times 4 matrix equal
\[
a_{11} = 3, \quad a_{21} = -7, \quad a_{12} = 1, \quad a_{14} = -11 \\
a_{13} = 5, \quad a_{14} = 19, \quad a_{23} = 78, \quad a_{24} = 61
\]

Write out the matrix.

Solution:
\[
\begin{bmatrix}
3 & -11 & 5 & 19 \\
-7 & 1 & 78 & 61
\end{bmatrix}
\]

Example 11: What is the size of each of the following matrices, and what does \(a_{32}\) equal in each matrix?

\[
A = \begin{bmatrix}
23 \\
11
\end{bmatrix}, \quad B = \begin{bmatrix}
9 & 7 \\
5 & 11 \\
23 & 41
\end{bmatrix}, \quad C = \begin{bmatrix}
9 & 0 & 21 \\
6 & 4 & 2 \\
31 & 14 & 53 \\
73 & 87 & 1
\end{bmatrix}
\]

Solution: The matrix \(A\) is a 2 \times 1 matrix and it has no entry in the third row, second column. \(B\) is 3 \times 2 matrix, and its entry in the third row, second column equals 41. The last matrix \(C\) is a 4 \times 3 matrix, and the entry in its third row and second column is 14.
Now that we’ve introduced matrices we can finish the notational scheme for a system of linear equations. Consider the following system of 2 equations in 2 unknowns.

\[ 3x_1 + 5x_2 = 17 \]
\[ -11x_1 + 13x_2 = 7. \]

There are two matrices associated with a linear system. They are the coefficient matrix and the augmented matrix. The coefficient matrix of this particular system is

\[ A = \begin{bmatrix} 3 & 5 \\ -11 & 13 \end{bmatrix}, \]

and the augmented matrix is

\[ \hat{A} = \begin{bmatrix} 3 & 5 & 17 \\ -11 & 13 & 7 \end{bmatrix}. \]

The system of equations consists of two equations in two unknowns. Thus, the coefficient matrix is a 2 \( \times \) 2 matrix, and the augmented matrix is a 2 \( \times \) 3 matrix.

**Example 12:** For the system of equations below, write out the coefficient matrix and the augmented matrix.

\[ 3x_1 - 5x_2 + x_4 = 23 \]
\[ x_1 - x_2 - x_3 + x_4 = 55 \]
\[ 15x_1 + 2x_3 + 5x_4 = 99. \]

Solution: If \( A \) denotes the coefficient matrix, and \( \hat{A} \) the augmented matrix, then \( A \) must be a 3 \( \times \) 4 matrix and \( \hat{A} \) is a 3 \( \times \) 5 matrix. They are shown below.

\[ A = \begin{bmatrix} 3 & -5 & 0 & 1 \\ 1 & -1 & -1 & 1 \\ 15 & 0 & 2 & 5 \end{bmatrix} \quad \text{and} \quad \hat{A} = \begin{bmatrix} 3 & -5 & 0 & 1 & 23 \\ 1 & -1 & -1 & 1 & 55 \\ 15 & 0 & 2 & 5 & 99 \end{bmatrix}. \]

**Example 13:** If the augmented matrix of a linear system of equations equals

\[ \begin{bmatrix} 3 & 0 & 2 & 8 \\ 0 & 1 & 1 & -4 \\ 2 & -1 & 0 & 23 \end{bmatrix} \]

What is the system of equations?

Solution: Since the augmented matrix has 3 rows, the system consists of 3 equations, and there are 3 unknowns associated with this system as the augmented matrix has 4 columns. The number of columns being one more than the number of unknowns. The system is shown below.

\[ 3x_1 + 2x_3 = 8 \]
\[ x_2 + x_3 = -4 \]
\[ 2x_1 - x_2 = 23 \]
Summary of Notational Scheme

1. The unknown variables in a system of equations are labeled $x_i$, where the index $i$ denotes which variable we are looking at. The largest value of $i$ tells us how many unknown variables are in the system. Thus, if our system has $n$ unknowns, the index $i$ takes on the values $1, 2, \ldots, n$.

2. The coefficients of the variables $x_i$ are labeled $a_{ij}$. The first subscript tells us in which equation $a_{ij}$ appears, and the second subscript tells us which of the unknown variables $x_j$, $a_{ij}$ multiplies. If our system has $m$ equations and $n$ unknowns, then the first subscript $i$, takes on the values $1, 2, \ldots, m$; the second subscript takes on the values $1, 2, \ldots, n$.

3. The constant value which appears on the right hand side of each equation is labeled $b_i$, where again the subscript tells us for which of the equations $b_i$ is the constant term. If there are $m$ equations, then the subscript $i$ takes on the values $1, 2, \ldots, m$.

4. To each system of $m$ linear equations in $n$ unknowns we associate two matrices. The $m \times n$ coefficient matrix whose entries are the coefficients of the unknowns. The second matrix is the augmented matrix, which is an $m \times (n + 1)$ matrix, whose first $n$ columns are the same as the columns of the coefficient matrix, and whose last column consists of the right hand sides of the equations in the system.

Example 14: Write out the coefficient matrix for the system

\[-5x_1 + 3x_2 = 4\]
\[7x_1 - 8x_2 = 1.\]

Solution: The coefficient matrix for this system equals

\[
\begin{bmatrix}
-5 & 3 \\
7 & -8
\end{bmatrix}.
\]

Example 15: What is the augmented matrix for the system of linear equations in the last example?

Solution:

\[
\begin{bmatrix}
-5 & 3 & 4 \\
7 & -8 & 1
\end{bmatrix}
\]

Example 16: What are the coefficient matrix and augmented matrix of the system below?

\[11x - 5y = 9\]

Solution:

\[
\begin{align*}
\text{coefficient matrix} &= \begin{bmatrix} 11 & -5 \end{bmatrix} \\
\text{augmented matrix} &= \begin{bmatrix} 11 & -5 & 9 \end{bmatrix}
\end{align*}
\]
Example 17: The matrix below is the coefficient matrix for a system of linear equations. How many equations and unknowns are in the system?

\[
\begin{bmatrix}
9 & 0 & -6 \\
2 & 4 & 17 \\
-3 & 5 & -32 \\
18 & 3 & 6
\end{bmatrix}
\]

Solution: The given coefficient matrix has 4 rows. Thus, there are 4 equations in the system. Since the coefficient matrix has 3 columns, one for each variable, there are 3 unknowns.

Example 18: What are the sizes of the coefficient matrix and augmented matrix for the following system of equations?

\[
\begin{align*}
3x_1 + x_2 + x_5 &= 2 \\
-x_1 + x_3 - x_4 &= 18.
\end{align*}
\]

Solution: Since there are 2 equations and 5 unknowns, the coefficient matrix is a $2 \times 5$ matrix, and the augmented matrix is a $2 \times 6$ matrix.